Elliptical distribution & Copula

Exercise 3.5.1 ~ 3.5.4. Simple example of Copula (Page 23 of Lecture 5)

$\boxed{\text{Notes about standard basis } \{e_i\}_{i=1}^{m}}$

$\forall\ x \in \mathbb{R}^m.\quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \quad x = x_1 e_1 + x_2 e_2 + \cdots + x_m e_m$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ \cdots\ e_m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$I_m x = x \quad \Rightarrow \quad e_i^T x = x_i$

$I_m = (e_1\ e_2 \cdots e_m) = \begin{pmatrix} e_1^T \\ e_2^T \\ \vdots \\ e_m^T \end{pmatrix} \qquad I_m x = \begin{pmatrix} e_1^T \\ e_2^T \\ \vdots \\ e_m^T \end{pmatrix} x = \begin{pmatrix} e_1^T x \\ e_2^T x \\ \vdots \\ e_m^T x \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$

Exercise 3.5.1. *Suppose that* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ *satisfy* $\|\mathbf{x}\| = \|\mathbf{y}\|$. *Show that there exists an orthogonal matrix* U *such that* $\mathbf{y} = U\mathbf{x}$. *Conclude that if* $f(\mathbf{x}) = f(U\mathbf{x})$ *for all* $U \in O(m)$ *then* f *depends on* $\mathbf{x}$ *only through its norm* $\|\mathbf{x}\|$. *Hint: Consider an orthonormal basis with* $\mathbf{x}/\|\mathbf{x}\|$ *as one of the basis vectors and another orthonormal basis with* $\mathbf{y}/\|\mathbf{y}\|$ *as one of the basis vectors.*

Consider orthogonal basis $\{\alpha_i\}_{i=1}^{m} = \{\alpha_1 = \frac{x}{\|x\|}, \alpha_2, \cdots \alpha_m\}$

another $\cdots\cdots\cdots$ $\{\beta_i\}_{i=1}^{m} = \{\beta_1 = \frac{y}{\|y\|}, \beta_2 \cdots \beta_m\}$

Method 1. By Orthogonal basis transformation theorem, there exists $U \in O(m)$. s.t $\beta_i = U\alpha_i,\ i = 1, \cdots m$.

Method 2. $A = (\alpha_1, \cdots \alpha_m)$. $B = (\beta_1 \cdots \beta_m)$, $A \cdot B \in O(m)$.

$B = B \cdot I_m = B \cdot A^T A$ choose $U = BA^T$. $U^T U = AB^T BA^T = I_m$.

$\Rightarrow U \in O(m)$

$\Rightarrow \beta_i = U\alpha_i,\ i = 1, \cdots m.$

$\frac{y}{\|y\|} = U \frac{x}{\|x\|},\quad \|x\| = \|y\| \quad \Rightarrow \quad y = Ux.$

Part 2. Goal: if $\|x\| = \|y\|$, then $f(x) = f(y)$

if $\|x\| = \|y\|$. by Part 1. there exists $U \in O(m)$ s.t $y = Ux$.

$f(y) = f(Ux) = f(x)$.

Exercise 3.5.2. *Show that if* X *has spherical distribution, then* $\mathbb{E}X = 0$ *and* $\mathrm{var}(X) = cI_m$ *for some* $c \geq 0$.

$X \overset{d}{=} UX,\ \forall\ U \in O(m)$.

$E(X) = E(UX)$

① Take $U = -I_m$. $E(UX) = E(-X) = -E(X) = E(X) \Rightarrow E(X) = 0$ vector.

② $Var(UX) = Var(X)$.

Choose $U = \begin{pmatrix} -e_1^T \\ e_2^T \\ \vdots \\ e_m^T \end{pmatrix}$ $\quad UX = \begin{pmatrix} -e_1^T X \\ e_2^T X \\ \vdots \\ e_m^T X \end{pmatrix} = \begin{pmatrix} -X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix}$ $\quad Cov(X_1, X_j) = Cov(-X_1, X_j) \quad \forall j \neq 1$.

$\Rightarrow Cov(X_1, X_j) = 0. \quad \forall j \neq 1$

Similarly. $U = \begin{pmatrix} e_1^T \\ \vdots \\ -e_i^T \\ \vdots \\ e_m^T \end{pmatrix} \to$ replacing e_i^T with $-e_i^T$ in I_m.

$UX = \begin{pmatrix} X_1 \\ \vdots \\ -X_i \\ \vdots \\ X_m \end{pmatrix} \Rightarrow Cov(X_i, X_j) = Cov(-X_i, X_j) = -Cov(X_i, X_j)$

$Cov(X_i, X_j) = 0. \quad \forall i \neq j$.

$\Rightarrow$ off-diagonals of $Var(X) = 0$.

Choose permutation matrix.

$U = \begin{pmatrix} e_2^T \\ e_1^T \\ \vdots \\ e_m^T \end{pmatrix} \quad UX = \begin{pmatrix} X_2 \\ X_1 \\ \vdots \\ X_m \end{pmatrix} \quad Var(X_1) = Var(X_2)$

Similarly. $U = \begin{pmatrix} e_1^T \\ \vdots \\ e_j^T \\ \vdots \\ e_i^T \\ e_m^T \end{pmatrix} \to$ switch the order of e_i^T & e_j^T in I_m.

$UX = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_j \\ \vdots \\ X_i \\ \vdots \\ X_m \end{pmatrix} \Rightarrow Cov(UX) = Cov(X)$ $\quad \Rightarrow$ diagonals are equal $= c \geq 0$.

$Var(X_i) = Var(X_j) \quad \forall i \neq j$. $\quad Var(X) = c I_m. \quad c \geq 0$.

Exercise 3.5.3. *Suppose* $X = \frac{1}{\sqrt{\tau}} Z$ *with* $Z \sim N(0, I_m)$, *where* $\tau > 0$ *is a random variable independent of* Z. *Show that* $\mathbb{E}X = 0$ *and* $\mathrm{var}(X) = \mathbb{E}[\tau^{-1}] I_m$.

Law of total expectation.

$E(X) = E(E(X|\tau)) = E(E(\frac{1}{\sqrt{\tau}} Z | \tau)) = E(\frac{1}{\sqrt{\tau}} E(Z|\tau))$ ind

$= E(\frac{1}{\sqrt{\tau}} E(Z)) = E(0) = 0$.

Law of total variance.

$Var(X) = Var(E(X|\tau)) + E(Var(X|\tau))$

$= Var(0) + E(Var(\frac{1}{\sqrt{\tau}} Z|\tau)) = E((\frac{1}{\sqrt{\tau}})^2 Var(Z|\tau))$ ind

$= E(\frac{1}{\tau} I_m) = E(\frac{1}{\tau}) I_m \to E(\frac{1}{\tau}) \neq 1/E(\tau)$.

Exercise 3.5.4. *If Z has a spherical distribution show that*

1. *The characteristic function* $\psi_Z(\mathbf{t})$ *is of the form* $\phi(\|\mathbf{t}\|^2)$ *for some* ϕ.
2. *Show that each component has characteristic function* $\psi_i(s) = \phi(s^2)$.
3. *Show that if one component of Z is Gaussian then Z is Gaussian.*

1. $UZ \overset{d}{=} Z.\ \forall U \in O(m).$

$\psi_{UZ}(t) = \psi_Z(t);\quad \psi_{UZ}(t) = E(e^{it^TUZ}) = E(e^{i(U^Tt)^TZ}) = \psi_Z(U^Tt)$
$= \psi_Z(t).$

$\Rightarrow \psi_Z(U^Tt) = \psi_Z(t).\ \forall U \in O(m).$

By Ex 3.5.1 $\Rightarrow \psi_Z(t) = h(\|t\|) = h(\sqrt{\|t\|^2}) = \phi(\|t\|^2).$

2. $\psi_{Z_i}(s) = E(e^{isZ_i}) = E(e^{it^TZ})$ take $t = se_i \leftarrow$ i-th standard basis.

$= \phi(\|t\|^2) = \phi(\|se_i\|^2) = \phi(s^2)$

3. W.L.O.G. Assume $Z_1 \sim N(0, \sigma^2) \longrightarrow$ By Ex 3.5.2 $E(Z) = 0.$

$\psi_{Z_1}(s) = e^{-\frac{1}{2}\sigma^2 s^2} \leftarrow N(0,\sigma^2)$

$= \phi(s^2)$

$\psi_Z(t) = \phi(\|t\|^2) = e^{-\frac{1}{2}\sigma^2\|t\|^2} = e^{-\frac{1}{2}\sigma^2 t^T t} \longleftarrow N_m(0, \sigma^2 I_m).$

$Z \sim N_m(0, \sigma^2 I_m)$

Understand the copula: Another look at dependence.

Mathematically $\Rightarrow$ variable transformation.

Example: $X \sim F_X(x),\ Y = 2X = g(X).\quad g(x) = 2x\quad g^{-1}(y) = \frac{y}{2}.$

$F_Y(y) = P(Y \le y) = P(2X \le y) = P(X \le \frac{y}{2}) = F_X(\frac{y}{2}).$

$F_Y(\cdot)$ can be represented by $F_X(\cdot)$ & $g^{-1}(\cdot)$

Similarly. $F_X(\cdot)$ . . . . $F_Y(\cdot)$ & $g(\cdot)$.

Copula: $X \sim F_X(x).\quad U = F_X(X).\ \sim U[0,1].$

$F_X(\cdot)$ can be represented by U & $F_X(\cdot)$ (or $F_X^{-1}(\cdot)$)

Compute the copula

- Joint CDF:

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0, \\ x^2y^2 & 0 \le x, y \le 1, \\ 1 & x > 1 \text{ and } y > 1, \\ \min(x^2, y^2) & \text{otherwise.} \end{cases}$$

$$C(u,v) = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)).$$

$$F_X(x) = \begin{cases} 0. & x < 0. \\ x^2 & 0 \le x \le 1 \\ 1. & x > 1. \end{cases}$$

Case 1. $0 \le y \le 1$

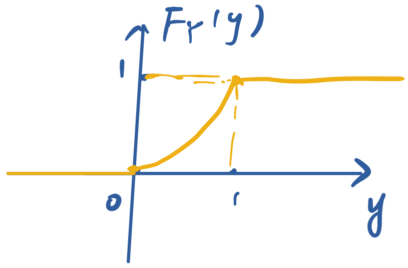
$F_{X,Y}(x,y) = x^2y^2$.

Case 2. $y > 1$

$F_{X,Y}(x,y) = \min(x^2, y^2) = x^2$

Similarly

$$F_Y(y) = \begin{cases} 0. & y < 0 \\ y^2 & 0 \le y \le 1 \\ 1. & y > 1. \end{cases}$$



$F_Y^{-1}(\cdot)$ exists

$0 \le y \le 1$

$F_Y^{-1}(y) = \sqrt{y}$.

$$C(u,v) = (F_X^{-1}(u))^2 (F_Y^{-1}(v))^2$$

$$= (\sqrt{u})^2 (\sqrt{v})^2$$

$$= uv. \quad 0 \le u, v \le 1.$$