**Exercise 4.8.6.** *Show that the principal components are uncorrelated by computing the covariance matrix of the transformed variables.*

$X \sim (\mu, \Sigma)$, $X \in \mathbb{R}^m$

$U = [u_1, u_2, \cdots u_m]$, $U \in \mathbb{R}^{m\times m}$, $\|u_i\| = 1$, $u_i^T u_j = 0 \ (i \neq j)$

$$Z = U^T X = \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix}$$

$$\begin{aligned} Cov(Z) &= Cov(U^T X, U^T X) \\ &= U^T Cov(X) U \\ &= U^T \Sigma U \qquad (\Sigma = U \Lambda U^T) \\ &= U^T (U \Lambda U^T) U \qquad (U^T U = I) \\ &= \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix} \end{aligned}$$

$\forall z_i, z_j \ (i \neq j)$, $cov(z_i, z_j) = 0$

**Exercise 4.8.19.** *Show that the matrix $AB$ with $A \in \mathbb{R}^{k\times l}$ and $B \in \mathbb{R}^{l\times m}$ has rank $\leq l$.*

Method 1: $Col(AB) \subseteq Col(A)$

$$\begin{aligned} rank(AB) &= dim(Col(AB)) \\ &\leq dim(Col(A)) \\ &= rank(A) \\ &\leq min(k, l) \\ &\leq l \end{aligned}$$

Method 2: $Row(AB) \subseteq Row(B)$

$$\begin{aligned} rank(AB) &= dim(Row(AB)) \\ &\leq dim(Row(B)) \\ &= rank(B) \leq min(l, m) \leq l \end{aligned}$$