STA 437/2005

Midterm Exam 2, Section LEC0201

Date: March 14, 2025

STA 437/2005, Midterm 2

Student Name: ____

(Please use capital letters)

Student Number: ____

I am taking (please circle): **STA437** **STA2005**

Instructions:

- Fill out your name and student number on this page.
- Carefully edit your answer in the provided space. Use the last two pages for your own notes (this will not be graded).
- Show all your work to receive full credit.
- This exam has three problems. Each problem is on a separate page.
- No electronic devices, notes, or books are allowed during the exam.

Good Luck!

Problem 1 (20 points)

Let $X \sim N_3(\mathbf{0}, \Sigma)$, where the covariance matrix is given by

$$\Sigma = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. (10 pts) What is the PCA transformation Z of X in this case?
   We could compute the eigenvalues and eigenvectors of Σ using the usual procedure. In this case we can be a bit quicker. Note that there is one eigenvactor $\mathbf{u}_3 := (0, 0, 1)$ with eigenvalue 1 (which can be seen immediately). All other eigenvectors are of the form $(*, *, 0)$ so enough to check the eigenvectors of the upper 2×2 submatrix. We have

   $$\det(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}) = 0 \text{ if and only if } \lambda = 1, 3.$$

   The associated eigenvectors are $\mathbf{u}_2 := \frac{1}{\sqrt{2}}(0, 1, -1)$, $\mathbf{u}_1 := \frac{1}{\sqrt{2}}(0, 1, 1)$ respectively. So the transformation is $Z = U^\top X$ where U has columns $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

2. (10 pts) Find the distribution of (Z_1, Z_2).
   Since $\Sigma = U \Lambda U^T$, the distribution of Z is $N_2(\mathbf{0}, \text{diag}(3, 1))$.

Problem 2 (20 points)

Consider the random vector $X = (X_1, X_2)$, where X is uniformly distributed on the unit square in $\mathbb{R}^2$, i.e., the probability density function of X is

$$f(x_1, x_2) = \begin{cases} \frac{1}{4}, & \text{if } -1 \le x_1, x_2 \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Answer the following questions.

1. (6 pts) Argue whether X follows spherical distribution.
   No. Z is uniformly distributed on the unit square, its distribution is not rotationally invariant, which means it does not follow the spherical distribution. Alternatively, one can also argue from the PDF that, the PDF of Z depends on the individual coordinate of Z, rather than simply the norm of Z.

2. (6 pts) Determine which of the following vectors follow the same distribution as X:

$$Y_1 = \begin{pmatrix} X_2 \\ X_1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} -X_1 \\ X_2 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} \frac{\sqrt{3}}{2}X_1 - \frac{1}{2}X_2 \\ \frac{1}{2}X_1 + \frac{\sqrt{3}}{2}X_2 \end{pmatrix}.$$

   Choose from the following options and briefly justify your answer:

   A. only Y_2.

   B. only Y_1 and Y_2.

   C. only Y_2 and Y_3.

   D. All of them.

   B, only Y_1 and Y_2. Since Z follows uniform distribution on the unit square, one can show that X_1 and X_2 are i.i.d. with marginal distribution $U[-1, 1]$. Therefore, it is invariant under permutation (Y_1)and reflection (Y_2), but not for a general rotation.

3. (8 pts) True or False (no need to provide justifications):

T / F: X_1, X_2 have the same distribution.

T / F: X_1, X_2 are independent.

T / F: $X_1 \sim U[-1, 1]$.

T / F: The copula of X_1 and X_2 is $C(u, v) = uv$.

Answer: T,T,T,T

Problem 3 (20 points)

Consider a vector $X = (X_1, X_2)$ whose distribution is a mixture of two Gaussian distributions with parameters: $\pi_1 = 0.6$, $\pi_2 = 0.4$ and

$$\mu_1 = (1, 2),\ \Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad \mu_2 = (0, 0),\ \Sigma_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

(i) (10 pts) Denote by $f_1(x)$, $f_2(x)$ the densities of the two Gaussian components. Suppose $f_1(3, 3) \approx 0.013$ and $f_2(3, 3) \approx 0.005$. Explain how you can use this information to compute the probability that the observation $(3, 3)$ comes from the first Gaussian component.

We have

$$p(z = 1|\mathbf{x} = (3, 3)) = \frac{\pi_1 p(\mathbf{x} = (3, 3)|z = 1)}{\pi_1 p(\mathbf{x} = (3, 3)|z = 1) + \pi_2 p(\mathbf{x} = (3, 3)|z = 2)}.$$

Since $p(\mathbf{x} = (3, 3)|z = 1) = 0.013$ and $p(\mathbf{x} = (3, 3)|z = 2) = 0.005$ we have everything we need to use this formula.

(ii) (10 pts) Compute $\mathbb{E}(X_1 X_2)$.

This follows by very standard arguments: First, $\mathbb{E}(X_1X_2) = \pi_1\mathbb{E}(X_1X_2|z = 1)+\pi_2\mathbb{E}(X_1X_2|z = 2)$. Second, $\mathbb{E}(X_1X_2|z = 1) = \text{cov}(X_1, X_2|z = 1) + \mathbb{E}(X_1|z = 1)\mathbb{E}(X_2|z = 1)$. Thus, we have

$$\mathbb{E}(X_1X_2) = 0.6 \cdot (0 + 1 \cdot 2) + 0.4 \cdot (1 + 0) = 1.6.$$

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