STA 437/2005

Midterm Exam 2, Section LEC0101

Date: March 14, 2025

STA 437/2005, Midterm 2

Student Name: ______________________________

(Please use capital letters)

Student Number: ______________________________

I am taking (please circle): **STA437** **STA2005**

Instructions:

- Fill out your name and student number on this page.
- Carefully edit your answer in the provided space. Use the last two pages for your own notes (this will not be graded).
- Show all your work to receive full credit.
- This exam has three problems. Each problem is on a separate page.
- No electronic devices, notes, or books are allowed during the exam.

Good Luck!

Problem 1 (20 points)

Let $X \sim N_3(\mathbf{0}, \Sigma)$, where the covariance matrix is given by

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

1. (10 pts) What is the PCA transformation Z of X in this case?
   We could compute the eigenvalues and eigenvectors of Σ using the usual procedure. In this case we can be a bit quicker. Note that there is one eigenvactor $\mathbf{u}_3 := (1, 0, 0)$ with eigenvalue 1. All other eigenvectors are of the form $(0, *, *)$ so enough to check the eigenvectors of the lower 2×2 submatrix. We have

   $$\det(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}) = 0 \text{ if and only if } \lambda = 1, 3.$$

   The associated eigenvectors are $\mathbf{u}_2 := \frac{1}{\sqrt{2}}(0, 1, -1)$, $\mathbf{u}_1 := \frac{1}{\sqrt{2}}(0, 1, 1)$ respectively. So the transformation is $Z = U^\top X$ where U has columns $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

2. (10 pts) Find the distribution of Z.
   Since $\Sigma = U \Lambda U^T$, the distribution of Z is $N_3(\mathbf{0}, \text{diag}(3, 1, 1))$.

Problem 2 (20 points)

Consider the random vector $X = (X_1, X_2)$, where X is uniformly distributed on the unit disk in $\mathbb{R}^2$, i.e., the probability density function of X is

$$f(x_1, x_2) = \begin{cases} \frac{1}{\pi}, & \text{if } x_1^2 + x_2^2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Answer the following questions.

1. (6 pts) Argue whether X follows spherical distribution.
   Yes. Since Z is uniformly distributed on the unit disk, its distribution is rotationally invariant, which means it follows the spherical distribution. Alternatively, one can also argue from the PDF that, the PDF of Z only depends on the norm of $\|X\| = \sqrt{X_1^2 + X_2^2}$, Specifically, $f(x_1, x_2)$ is constant on the unit disk and zero elsewhere.

2. (6 pts) Determine which of the following vectors follow the same distribution as Z:

$$Y_1 = \begin{pmatrix} X_2 \\ X_1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} -X_1 \\ X_2 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} \frac{\sqrt{3}}{2}X_1 - \frac{1}{2}X_2 \\ \frac{1}{2}X_1 + \frac{\sqrt{3}}{2}X_2 \end{pmatrix}.$$

   Choose from the following options and justify your answer:

   A. only Y_2.

   B. only Y_1 and Y_2.

   C. only Y_2 and Y_3.

   D. All of them.

   D, all of them. Since Z follows spherical distribution, it is invariant under permutation, reflection and rotation, which corresponds to Y_1, Y_2 and Y_3 respectively.

3. (8 pts) True or False (no need to provide justifications):

T / F: X_1, X_2 have the same distribution.

T / F: X_1, X_2 are independent.

T / F: $X_1 \sim U[-1, 1]$.

T / F: The copula of X_1 and X_2 is $C(u, v) = uv$.

T,F,F,F. Note that the copula is uv if and only if the variables are independent.

Problem 3 (20 points)

Consider a vector $X = (X_1, X_2)$ whose distribution is a mixture of two Gaussian distributions with parameters: $\pi_1 = 0.7$, $\pi_2 = 0.3$ and

$$\mu_1 = (1, 2),\ \Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad \mu_2 = (0, 0),\ \Sigma_2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

1. (10 pts) Suppose we get a i.i.d. sample of size 5 from the Bernoulli distribution with parameter 0.3 (probability of 1) and the sample is $(1, 1, 0, 1, 0)$ suppose we also have an i.i.d. sample from the bivariate standard normal:

$$\begin{bmatrix} 0.04 \\ 1.31 \end{bmatrix}, \begin{bmatrix} 0.98 \\ 0.88 \end{bmatrix}, \begin{bmatrix} 0.48 \\ 0.97 \end{bmatrix}, \begin{bmatrix} -0.81 \\ 0.28 \end{bmatrix}, \begin{bmatrix} -0.16 \\ 1.94 \end{bmatrix}.$$

   Without any explicit calculations, explain how you can use it to get a sample from the given mixture distribution.
   Remember how observations from a Gaussian Mixture model can be generated. We have a binary latent variable with $\mathbb{P}(Z = 0) = \pi_1$, $\mathbb{P}(Z = 1) = \pi_2$. Then we first generate Z and then if $Z = 0$, we generate an observation from the first Gaussian component and if $Z = 1$ we generate from the second Gaussian component. In this question the sample $(1, 1, 0, 1, 0)$ gives us a random sample of the latent variable. So for example to generate the first sample from the Gaussian mixture model we need to sample from the second Gaussian component. The sample $(0.04, 1.31)$ is from $N_2(0, I_2)$. To make it into the sample from the second component, we simply need to multiply it by 2 - obtaining $(0.08, 2.32)$. We continue like this - whenever we see zero, we produce a sample from the first Gaussian by adding to the corresponding 2d vector $(1, 2)$. Whenever we see one we produce a sample from the second Gaussian by multiplying the corresponding 2D vector by 2.

2. (10 pts) Compute $\mathbb{E}(X_1^2)$.
   This follows by very standard arguments: First, $\mathbb{E}(X_1^2) = \pi_1 \mathbb{E}(X_1^2 | z = 1) + \pi_2 \mathbb{E}(X_1^2 | z = 2)$. Second, $\mathbb{E}(X_1^2 | z = 1) = \text{var}(X_1 | z = 1) + \mathbb{E}(X_1 | z = 1)^2$. We have $\mathbb{E}(X_1^2) = 0.7 \cdot (1 + 1^2) + 0.3 \cdot (4 + 0^2) = 2.6$

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