STA 437/2005, Midterm 1

Instructions:

- Fill out your name and student number on this page.
- Carefully edit your answer in the provided space. Use the last two pages for your own notes (this will not be graded).
- Show all your work to receive full credit.
- This exam has three problems. Each problem is on a separate page.
- No electronic devices, notes, or books are allowed during the exam.

Good Luck!

Problem 1 (20 pts)

Consider the singular value decomposition of matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix},$$

where $\mathbf{U} = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $\mathbf{V} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

1. (5 pts) What is the rank of $\mathbf{A}$? Justify your answer.

   The rank is two. This is the number of nonzero singular values. Equivalently: The rank is not affected by multiplying from the left or from the right by invertible matrices and so it is the same as the rank of D.

2. (5 pts) What is the spectral decomposition of $\mathbf{A}^\top\mathbf{A}$?

   $V(D^\top D)V^\top$, where
   $$D^\top D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

3. (5 pts) What is the spectral decomposition of $(\mathbf{A}^\top\mathbf{A})^{1/2}$?

   $V\tilde{D}V^\top$, where
   $$\tilde{D} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$$

4. (5 pts) Compute $\mathbf{U}^\top\mathbf{U}$.

   This is the identity matrix!

Problem 2 (20 pts)

Let $X_1, X_2 \sim N_m(\mu, \Sigma)$ be independent. Consider the random vector $Z = \begin{pmatrix} 2X_1 + X_2 \\ X_1 - 2X_2 \end{pmatrix}$, answer the following questions:

- (12 pts) Compute the mean vector and covariance matrix of Z;

  This problem is very similar to section 101.

- (8 pts) Is $2X_1 + X_2$ independent of $X_1 - 2X_2$?

Problem 3 (20 points)

Consider the random variables $H \sim \mathcal{N}(\eta, 1)$ and $X_1, X_2 \sim \mathcal{N}(\mu, 1)$, where all three are independent to each other. Let us define the random bivariate vector Y such that $Y_1 = X_1 + a_1 H$ and $Y_2 = X_2 + a_2 H$, for $a_1, a_2 \in \mathbb{R}$.

(i) (6 pts) Derive the distribution of the vector Y, specifying its mean and covariance matrix.

By independence, (X_1, X_2, H) are jointly Gaussian and so it (Y_1, Y_2). We have

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ H \end{bmatrix} \quad \text{and so} \quad \mathbb{E}(\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}) = \begin{bmatrix} \mu + a_1\eta \\ \mu + a_2\eta \end{bmatrix}$$

Similarly

$$\text{var}(\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}) = \begin{bmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \end{bmatrix} I_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 1 + a_1^2 & a_1 a_2 \\ a_1 a_2 & 1 + a_2^2 \end{bmatrix}.$$

(ii) (6 pts) Derive the conditional distribution $Y \mid H = h$.

Note that

$$\begin{aligned}
\mathbb{E}(Y_1|H) &= \mathbb{E}(X_1|H) + a_1 H = \mu + a_1 H \\
\mathbb{E}(Y_2|H) &= \mathbb{E}(X_2|H) + a_2 H = \mu + a_2 H \\
\text{var}(Y_1|H) &= \text{var}(X_1 + a_1 H|H) = \text{var}(X_1|H) = \text{var}(X_1) = 1 \\
\text{var}(Y_2|H) &= \text{var}(X_2 + a_2 H|H) = \text{var}(X_2|H) = \text{var}(X_2) = 1 \\
\text{cov}(Y_1, Y_2|H) &= \text{cov}(X_1 + a_1 H, X_2 + a_2 H|H) = \text{cov}(X_1, X_2|H) = \text{cov}(X_1, X_2) = 0
\end{aligned}$$

(iii) (8 pts) Consider the joint distribution of (Y_1, Y_2, H). Without calculating the inverse of its covariance matrix explicitly, argue what is its $(1, 2)$ entry.

We see above that Y_1, Y_2 are conditionally independent given H. So the $(1, 2)$ entry will be zero.

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