STA 437/2005, Midterm 1

Instructions:

- Fill out your name and student number on this page.
- Carefully edit your answer in the provided space. Use the last two pages for your own notes (this will not be graded).
- Show all your work to receive full credit.
- This exam has three problems. Each problem is on a separate page.
- No electronic devices, notes, or books are allowed during the exam.

Good Luck!

Problem 1 (20 points)

Consider the spectral decomposition of matrix $\mathbf{A}$ as

$$\mathbf{A} = \begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}.$$

1. (5 pts) What is the rank of $\mathbf{A}$? Justify your answer.

   The rank is two. The rank does not change if we multiply by an invertible matrix so it is the same as the middle matrix above.

2. (5 pts) Is $\mathbf{A}$ positive semi-definite? Justify your answer.

   No, it has a negative eigenvalue. This can be also argued from the definition. If $A = U\Lambda U^\top$ then take $\mathbf{x}$ such that $\mathbf{x} = U\mathbf{e}_1$. Then $\mathbf{x}^\top A\mathbf{x} = \mathbf{e}_1^\top \Lambda \mathbf{e}_1 = -1 < 0$.

3. (5 pts) What is the spectral decomposition of $\mathbf{A}^2$?

   $A^2 = U\Lambda^2 U^\top$

4. (5 pts) Is $\mathbf{A}^2$ positive semi-definite? Justify your answer.

   All eigenvalues are nonnegative, so yes. This could be also argued from the definition: for all $\mathbf{x}$, $\mathbf{x}^\top A^2 \mathbf{x} = \mathbf{y}^\top \Lambda^2 \mathbf{y} = y_1^2 + 4y_3^2 \geq 0$, where $\mathbf{y} = U^\top \mathbf{x}$

Problem 2 (20 points)

Let $X_1, X_2 \sim N_m(\mathbf{0}, \Sigma)$ be independent. Consider the random vector $Z = \begin{pmatrix} X_1 + X_2 \\ X_1 - X_2 \end{pmatrix}$, answer the following questions:

- (12 pts) Find the distribution of Z;

  Z is Gaussian. The mean is

  $$\mathbb{E}Z = \begin{bmatrix} \mathbb{E}(X_1 + X_2) \\ \mathbb{E}(X_1 - X_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

  For the covariance note that $\operatorname{var}(X_1 + X_2) = \operatorname{var}(X_1) + \operatorname{var}(X_2) = 2\Sigma$ (by independence). SImilarly, $\operatorname{var}(X_1 - X_2) = \operatorname{var}(X_1) + \operatorname{var}(X_2) = 2\Sigma$ and

  $$\operatorname{cov}(X_1 + X_2, X_1 - X_2) = \Sigma - 0 + 0 - \Sigma = 0.$$

  And so the covariance matrix of Z is

  $$\begin{bmatrix} 2\Sigma & 0 \\ 0 & 2\Sigma \end{bmatrix}.$$

- (8 pts) Argue whether $X_1 + X_2$ is independent of $X_1 - X_2$?

  As we just calculated, the covariance matrix between these two vectors is zero and so they are independent.

Problem 3 (20 points)

Consider the following thre-dimensional multivariate Gaussian

$$X \sim N_3(\mu, \Sigma), \quad \text{where } \mu = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 3/2 & 1/2 & 1 \\ 1/2 & 3/2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

(i) (8 pts) Find the conditional distribution of the subvector $(X_1, X_2) \mid X_3$.

Let $A = \{1, 2\}$, $B = \{3\}$. The conditional mean is

$$\mu_A + \Sigma_{A,B}\Sigma_{B,B}^{-1}(X_B - \mu_B) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tfrac{1}{2}(X_3 - 3) = \begin{bmatrix} \frac{1}{2}(X_3 + 5) \\ \frac{1}{2}(X_3 - 3) \end{bmatrix}$$

The conditional covariance is

$$\Sigma_{A,A} - \Sigma_{A,B}\Sigma_{B,B}^{-1}\Sigma_{B,A} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(ii) (6 pts) Without calculating the inverse of the covariance Σ^{-1} explicitly, argue what will be the value of the $(1, 2)$ entries of Σ^{-1}.

The calculations above show that X_1, X_2 are independent conditionally on X_3. As a result, the $(1, 2)$ entry of Σ^{-1} must be zero.

(iii) (6 pts) Define $Y_1 = X_1 + X_2$, $Y_2 = X_2 + X_3$ and $Y_3 = X_3 + X_1$. Derive the distribution of (Y_1, Y_2).

Since this distribution is Gaussian. It is enough to compute the mean and the covariance matrix. Note that

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \text{and so} \quad \mathbb{E}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

and

$$\operatorname{var}(\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3/2 & 1/2 & 1 \\ 1/2 & 3/2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & \frac{11}{2} \end{pmatrix}$$

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