STA 414/2104:
Statistical Methods of Machine Learning II

Week 11: Neural Networks and Optimization
(I am sorry for moving online this week)

Piotr Zwiernik

University of Toronto

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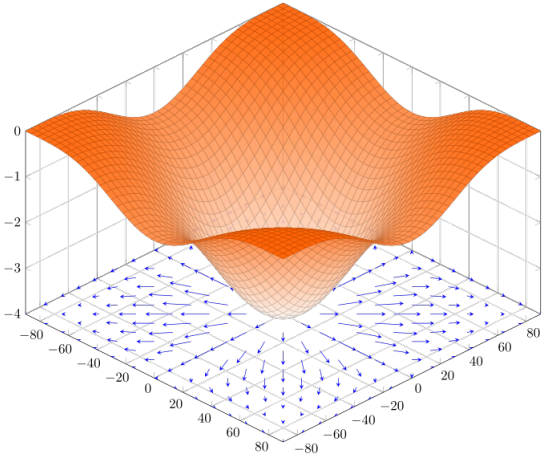
Basics of Optimization in ML

Gradients

Differentiable function $f : \mathbb{R}^d \to \mathbb{R}$,
$\mathbf{w} = (w_1, \ldots, w_d)$,
gradient of f at w

$$\nabla f(\mathbf{w}) = \begin{bmatrix} \frac{\partial f}{\partial w_1}(\mathbf{w}) \\ \vdots \\ \frac{\partial f}{\partial w_d}(\mathbf{w}) \end{bmatrix}$$

$$f(\mathbf{w} + \eta \mathbf{u}) \approx f(\mathbf{w}) + \eta \nabla f(\mathbf{w})^\top \mathbf{u}$$



Important geometric interpretation of the gradient

The gradient gives the direction of the steepest local increase of f.

What is optimization?

- Typical setup (in machine learning, other areas):
  - Formulate a problem
  - Design a solution (usually a model)
  - Use some quantitative measure to determine how good the solution is.
- E.g., classification:
  - Create a system to classify images
  - Model is some classifier, like the logistic regression
  - Quantitative measure is misclassification error (lower is better in this case)
- In almost all cases, you end up with a loss minimization problem of the form

$$\text{minimize}_{\mathbf{w}}\ E(\mathbf{w})$$

- Example: Least squares

$$\text{minimize}\ E(\mathbf{w}) = \tfrac{1}{2}\sum_{n=1}^{N}(\mathbf{x}_n^\top \mathbf{w} - t_n)^2.$$

Error minimization

- Training an ML model always reduces to solving an optimization problem

$$\text{minimize}_{\mathbf{w}}\ E(\mathbf{w}), \qquad \mathbf{w}^* := \arg\min_{\mathbf{w}}\ E(\mathbf{w}).$$

- Standard approach is gradient descent $\mathbf{w}^{t+1} = \mathbf{w}^t - \eta \nabla E(\mathbf{w}^t)$, where $\eta \in (0, 1]$ is the step size (aka learning rate) with w^0 some initial point.

- For the least squares, $\text{minimize}\ E(\mathbf{w}) = \frac{1}{2}\sum_{n=1}^{N}(\mathbf{x}_n^\top \mathbf{w} - t_n)^2$ we have

$$\nabla E(\mathbf{w}) = \sum_{n=1}^{N} \mathbf{x}_n(\mathbf{x}_n^\top \mathbf{w} - t_n).$$

Gradient descent derivation

- Suppose we are at $\mathbf{w}$ and we want to pick a direction $\mathbf{u}$ such that $E(\mathbf{w} + \eta\mathbf{u})$ is smaller than $E(\mathbf{w})$ for a step size η, $\|\mathbf{u}\| = 1$.
- The first-order Taylor series approximation of $E(\mathbf{w} + \eta\mathbf{u})$ around $\mathbf{w}$ is:

$$E(\mathbf{w} + \eta\mathbf{u}) \;=\; E(\mathbf{w}) + \eta\nabla E(\mathbf{w})^\top \mathbf{u} + o(\eta) \;\approx\; E(\mathbf{w}) + \eta\nabla E(\mathbf{w})^\top \mathbf{u}.$$

- Direction u should have a negative inner product with $\nabla E(\mathbf{w})$, e.g. $-\frac{\nabla E(\mathbf{w})}{\|\nabla E(\mathbf{w})\|}$.
- This approximation gets better as η gets smaller.

How do we choose the step size in GD? $\mathbf{w}^{t+1} = \mathbf{w}^t - \eta\nabla E(\mathbf{w}^t)$

- Simple strategy: start with a big η and progressively make it smaller by e.g. halving it until the function decreases.
- The sequence of step sizes is referred to as learning rate schedule.

When did the GD converge?

- The vector $\mathbf{w}$ is a fixed point if $\nabla E(\mathbf{w}) = \mathbf{0}$.
- This is never possible in practice. So we stop iterations if gradient is smaller than a threshold, $\|\nabla E(\mathbf{w})\| < \tau$.
- If the function is convex then we have reached a global minimum.
- If the function is not convex, what did we obtain?
- Probably a local minimum or a saddle point.

Stochastic Gradient Descent

In most cases, we minimize an average over data points:

$$E(\mathbf{w}) = \frac{1}{N}\sum_{i=1}^{N} L(t_n, y(\mathbf{x}_n, \mathbf{w})), \qquad \nabla E(\mathbf{w}) = \frac{1}{N}\sum_{n=1}^{n} \nabla L(t_n, y(\mathbf{x}_n, \mathbf{w})),$$

which is hard to compute when N is very large.

At each iteration, use a sub-sample of data to estimate the gradient

$$\mathbf{w}^{t+1} = \mathbf{w}^t - \eta \frac{1}{|S|} \sum_{n \in S} \nabla L(t_n, y(\mathbf{x}_n, \mathbf{w})).$$

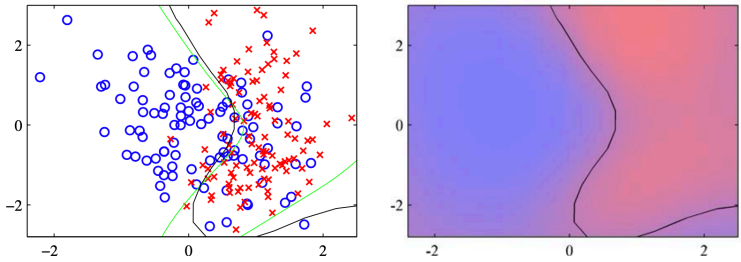
(Here $|S|$ denotes the number of elements in the set S. Standard SGD has $|S| = 1$)

ML terminology: Computing gradients using the full dataset is called batch learning, using subsets of data is called mini-batch learning.

Introducing neural networks

Limits of Linear Classification

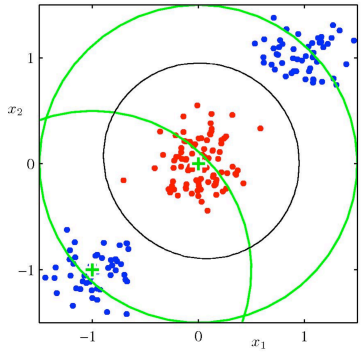
Many data sets are not linearly separable.



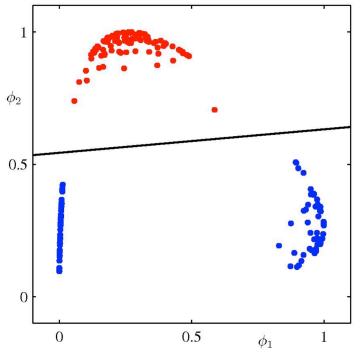
As a result, linear classification methods will not always work well.

Sometimes we may choose a suitable feature map

Original input space



Corresponding feature space using two Gaussian basis functions

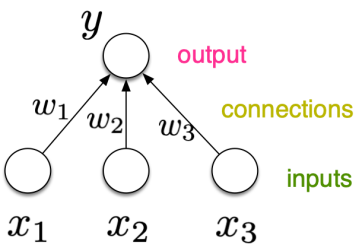


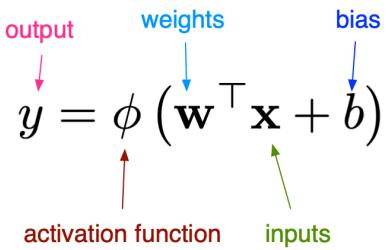
Motivating problem

Designing feature maps can be hard. Can we learn them automatically?

A Simpler Neuron

For neural nets, we use a simple model for neuron, or **unit**:

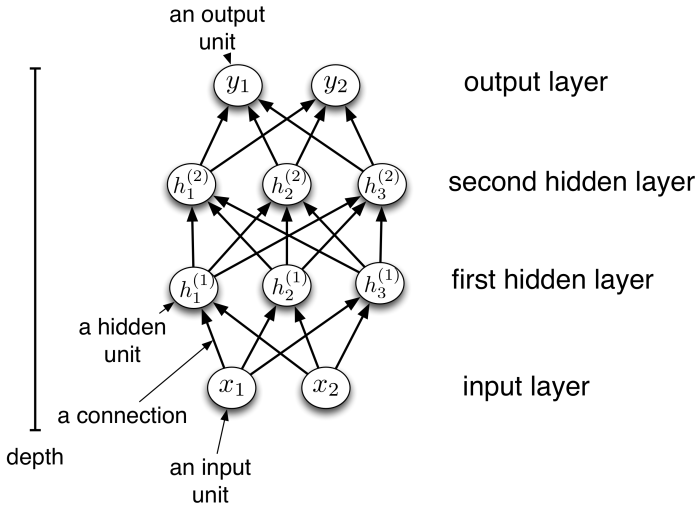




- Same as logistic regression: $y = \sigma(\mathbf{w}^\top \mathbf{x} + b)$
- By throwing together lots of these simple neuron-like processing units, we can do some powerful computations!

A Feed-Forward Neural Network

- A directed acyclic graph
- Units are grouped into layers

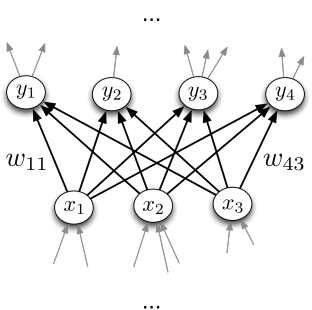


Multilayer Perceptrons

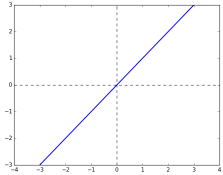
- A multi-layer network consists of fully connected layers.
- In a fully connected layer, all input units are connected to all output units.
- The outputs are a function of the input units:

$$\mathbf{y} = f(\mathbf{x}) = \phi(\mathbf{W}\mathbf{x} + \mathbf{b})$$

$\phi : \mathbb{R} \to \mathbb{R}$ is applied component-wise.

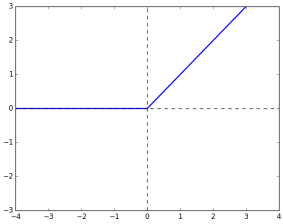


Some Activation Functions



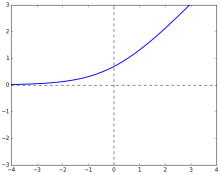
Identity

$\phi(z) = z$



Rectified Linear Unit (ReLU)

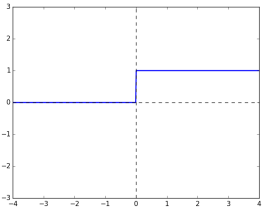
$\phi(z) = \max(0, z)$



Soft ReLU

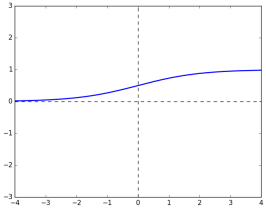
$\phi(z) = \log 1 + e^z$

More Activation Functions



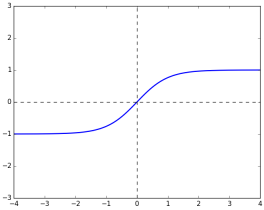
Hard Threshold

$$\phi(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z \leq 0 \end{cases}$$



Logistic

$$\phi(z) = \frac{1}{1 + e^{-z}}$$



Hyperbolic Tangent (tanh)

$$\phi(z) = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

Computation in Each Layer

Each layer computes a function.

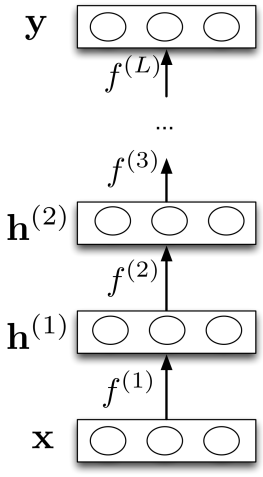
$$\mathbf{h}^{(1)} = f^{(1)}(\mathbf{x}) = \phi(\mathbf{W}^{(1)}\mathbf{x} + \mathbf{b}^{(1)})$$
$$\mathbf{h}^{(2)} = f^{(2)}(\mathbf{h}^{(1)}) = \phi(\mathbf{W}^{(2)}\mathbf{h}^{(1)} + \mathbf{b}^{(2)})$$
$$\vdots$$
$$\mathbf{y} = f^{(L)}(\mathbf{h}^{(L-1)})$$

The network computes a composition of functions.

$$\mathbf{y} = f^{(L)} \circ \cdots \circ f^{(1)}(\mathbf{x}).$$

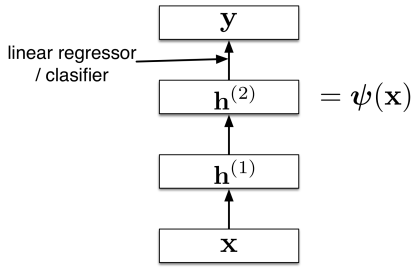
The last layer depends on the task.

- Regression: $\mathbf{y} = f^{(L)}(\mathbf{h}^{(L-1)}) = (\mathbf{w}^{(L)})^\top \mathbf{h}^{(L-1)} + b^{(L)}$
- Classification: $\mathbf{y} = f^{(L)}(\mathbf{h}^{(L-1)}) = \sigma\big((\mathbf{w}^{(L)})^\top \mathbf{h}^{(L-1)} + b^{(L)}\big)$

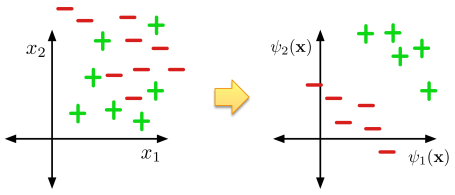


Feature Learning

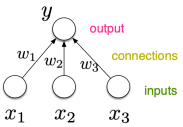
Neural nets can be viewed as a way of learning features:

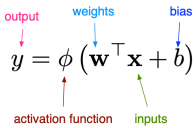


The goal:



Feature Learning



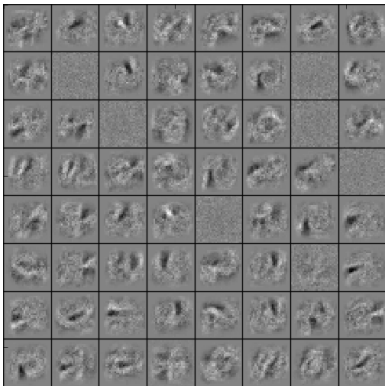


- Suppose we try to classify images of handwritten digits.
- Each image is represented as a vector of $28 \times 28 = 784$ pixel values.
- Each hidden unit in the first layer acts as a **feature detector**.
- We can visualize $\mathbf{w}$ by reshaping it into an image.

Below is an example that responds to a diagonal stroke.

Features for Classifying Handwritten Digits

Features learned by the first hidden layer of a handwritten digit classifier:



Unlike hard-coded feature maps (e.g., in polynomial regression), features learned by neural networks adapt to patterns in the data.

Expressive Power of Linear Networks

- Consider a linear layer: the activation function was the identity. The layer just computes an affine transformation of the input.
- Any sequence of linear layers is equivalent to a single linear layer.

$$\mathbf{y} = \underbrace{\mathbf{W}^{(3)}\mathbf{W}^{(2)}\mathbf{W}^{(1)}}_{\triangleq \mathbf{W}'}\mathbf{x}$$

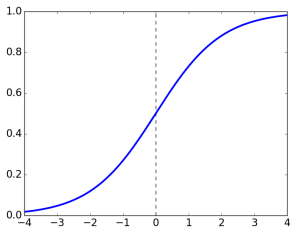
- Deep linear networks can only represent linear functions — no more expressive than linear regression.

Expressive Power of Non-linear Networks

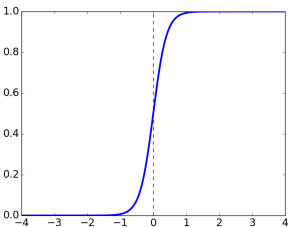
- Multi-layer feed-forward neural networks with non-linear activation functions
- **Universal Function Approximators**: They can approximate any function arbitrarily well.
- True for various activation functions (e.g. thresholds, logistic, ReLU, etc.)

Expressivity of the Logistic Activation Function

- What about the logistic activation function?
- Approximate a hard threshold by scaling up w and b.



$y = \sigma(x)$



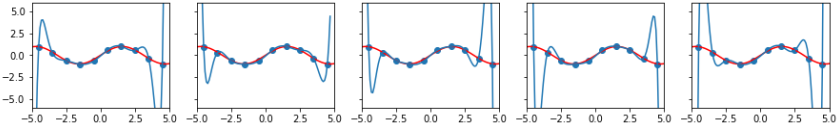
$y = \sigma(5x)$

- Logistic units are differentiable, so we can learn weights with gradient descent.

What is Expressivity Good For?

- May need a very large network to represent a function.
- Non-trivial to learn the weights that represent a function.
- If you can learn any function, over-fitting is potentially a serious concern!

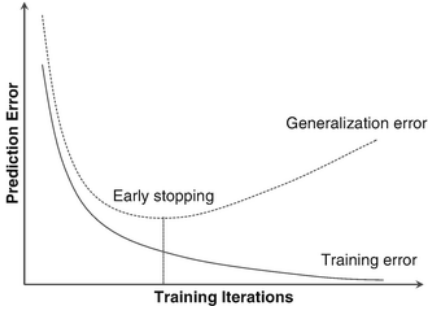
  For the polynomial feature mappings, expressivity increases with the degree M, eventually allowing multiple perfect fits to the training data. This motivated L^2 regularization.



- Do neural networks over-fit and how can we regularize them?

Regularization and Over-fitting for Neural Networks

- The topic of over-fitting (when & how it happens, how to regularize, etc.) for neural networks is not well-understood, even by researchers!
  - In principle, you can always apply L^2 regularization.
- A common approach is early stopping, or stopping training early, because over-fitting typically increases as training progresses.



- Benign overfitting is a heavily studied phenomenon.

Backpropagation

Learning Weights in a Neural Network

- Goal is to learn weights in a multi-layer neural network using gradient descent.
- Weight space for a multi-layer neural net: one set of weights for each unit in every layer of the network
- Define a loss $\mathcal{L}(t, y) = \mathcal{L}(t, y(x, \mathbf{w}))$ and compute the gradient of the cost

$$E(\mathbf{w}) \;=\; \tfrac{1}{N} \sum_{n=1}^{N} \mathcal{L}(t_n, y(x_n, \mathbf{w})),$$

  which is the average loss over all the training examples.
- How we can calculate $\nabla E(\mathbf{w})$ efficiently?

Example: Two-Layer Neural Network

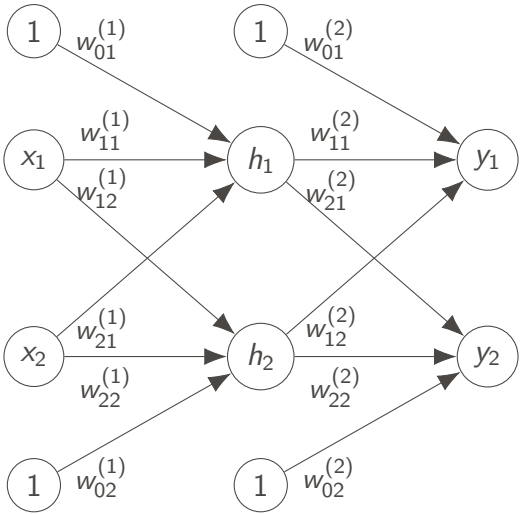
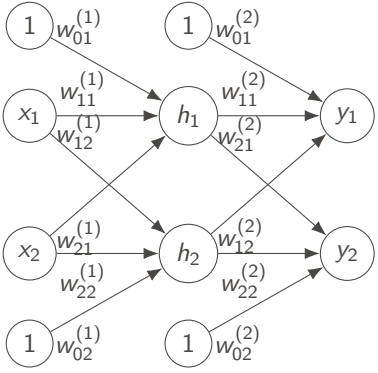


Figure 1: Two-Layer Neural Network

Computations for Two-Layer Neural Network

A neural network computes a composition of functions.



$$z_1^{(1)} = w_{01}^{(1)} \cdot 1 + w_{11}^{(1)} \cdot x_1 + w_{21}^{(1)} \cdot x_2$$

$$h_1 = \sigma(z_1)$$

$$z_1^{(2)} = w_{01}^{(2)} \cdot 1 + w_{11}^{(2)} \cdot h_1 + w_{21}^{(2)} \cdot h_2$$

$$y_1 = z_1$$

$$z_2^{(1)} =$$

$$h_2 =$$

$$z_2^{(2)} =$$

$$y_2 =$$

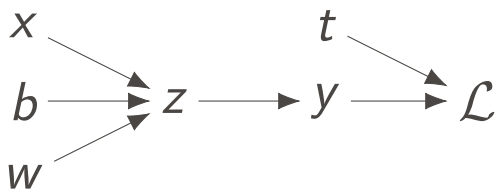
$$\mathcal{L} = \frac{1}{2}\left((y_1 - t_1)^2 + (y_2 - t_2)^2\right)$$

Simplified Example: Logistic Least Squares

$$z = wx + b$$

$$y = \sigma(z)$$

$$\mathcal{L} = \frac{1}{2}(y - t)^2$$



Computation Graph:

- The nodes represent the inputs and computed quantities.
- The edges represent which nodes are computed directly as a function of which other nodes.

Univariate Chain Rule

Let $z = f(y)$ and $y = g(x)$ be uni-variate functions.
Then $z = f(g(x))$.

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\,\frac{\mathrm{d}y}{\mathrm{d}x}$$

Logistic Least Squares: Gradient for w

Computing the loss:

$$z = wx + b$$
$$y = \sigma(z)$$
$$\mathcal{L} = \frac{1}{2}(y - t)^2$$

Computing the gradient for w:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial w} &= \frac{\partial \mathcal{L}}{\partial y}\,\frac{\partial y}{\partial w} \\ &= \frac{\partial \mathcal{L}}{\partial y}\,\frac{\partial y}{\partial z}\,\frac{\partial z}{\partial w} \\ &= (y - t)\;\sigma'(z)\;x \\ &= (\sigma(wx + b) - t)\sigma'(wx + b)x\end{aligned}$$

Logistic Least Squares: Gradient for b

Computing the loss:

$$z = wx + b$$
$$y = \sigma(z)$$
$$\mathcal{L} = \frac{1}{2}(y - t)^2$$

Computing the gradient for b:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial b} &= \frac{\partial \mathcal{L}}{\partial y}\,\frac{\partial y}{\partial b} \\ &= \frac{\partial \mathcal{L}}{\partial y}\,\frac{\partial y}{\partial z}\,\frac{\partial z}{\partial b} \\ &= (y - t)\;\sigma'(z)\;1 \\ &= (\sigma(wx + b) - t)\sigma'(wx + b)1\end{aligned}$$

Comparing Gradient Computations for w and b

Computing the loss:

$$z = wx + b$$
$$y = \sigma(z)$$
$$\mathcal{L} = \frac{1}{2}(y - t)^2$$

Computing the gradient for w:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial w} &= \frac{\partial \mathcal{L}}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial w} \\ &= (y - t)\,\sigma'(z)\,x\end{aligned}$$

Computing the gradient for b:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial b} &= \frac{\partial \mathcal{L}}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial b} \\ &= (y - t)\,\sigma'(z)\,1\end{aligned}$$

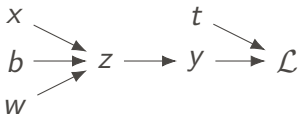
Drawbacks

- For larger networks these computations become cumbersome
- There will be many repeated terms, e.g. $\sigma'(z)$ appears on both sides.

Structured Way of Computing Gradients

Computing the loss:

$$z = wx + b$$
$$y = \sigma(z)$$
$$\mathcal{L} = \frac{1}{2}(y - t)^2$$



Computing the gradients:

$$\frac{\partial \mathcal{L}}{\partial y} = (y - t)$$
$$\frac{\partial \mathcal{L}}{\partial z} = \frac{\partial \mathcal{L}}{\partial y}\,\sigma'(z)$$

$$\frac{\partial \mathcal{L}}{\partial w} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}w} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}z}\,x \qquad \frac{\partial \mathcal{L}}{\partial b} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}b} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}z}\,1$$

Error Signal Notation

- Let $\overline{y}$ denote the derivative $\mathrm{d}\mathcal{L}/\mathrm{d}y$, called the error signal.
- Error signals are just values our program is computing (rather than a mathematical operation).

Computing the loss:

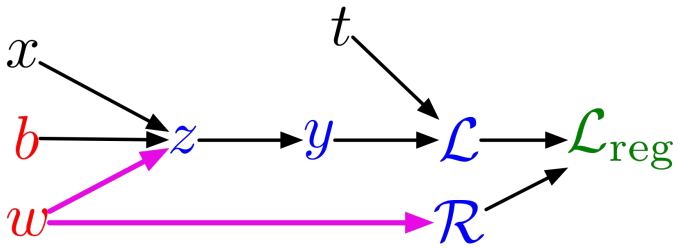
$$z = wx + b$$
$$y = \sigma(z)$$
$$\mathcal{L} = \frac{1}{2}(y - t)^2$$

Computing the derivatives:

$$\overline{y} = (y - t)$$
$$\overline{z} = \overline{y}\,\sigma'(z)$$
$$\overline{w} = \overline{z}\,x \qquad \overline{b} = \overline{z}$$

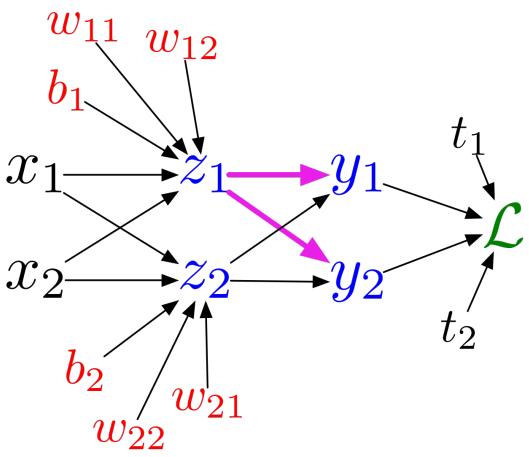
(previous slide: $\frac{\partial \mathcal{L}}{\partial y} = (y - t)$, $\frac{\partial \mathcal{L}}{\partial z} = \frac{\partial \mathcal{L}}{\partial y}\,\sigma'(z)$, $\frac{\partial \mathcal{L}}{\partial w} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}z}\,x$, $\frac{\partial \mathcal{L}}{\partial b} = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}z}\,1$)

Computation Graph has a Fan-Out > 1



$$z = wx + b$$
$$y = \sigma(z)$$
$$\mathcal{L} = \frac{1}{2}(y - t)^2$$
$$\mathcal{R} = \frac{1}{2}w^2$$
$$\mathcal{L}_{\text{reg}} = \mathcal{L} + \lambda\mathcal{R}$$

Computation Graph has a Fan-Out > 1



$$z_\ell = \sum_j w_{\ell j} x_j + b_\ell$$

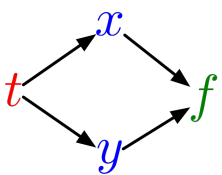
$$y_k = \frac{e^{z_k}}{\sum_\ell e^{z_\ell}}$$

$$\mathcal{L} = -\sum_k t_k \log y_k$$

Multi-variate Chain Rule

Suppose we have functions $f(x, y)$, $x(t)$, and $y(t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$



Example:

$$f(x, y) = y + e^{xy}$$
$$x(t) = \cos t$$
$$y(t) = t^2$$

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= (ye^{xy}) \cdot (-\sin t) + (1 + xe^{xy}) \cdot 2t\end{aligned}$$

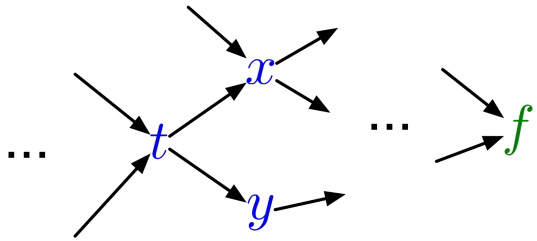
Multi-variate Chain Rule

In the context of back-propagation:

Mathematical expressions to be evaluated

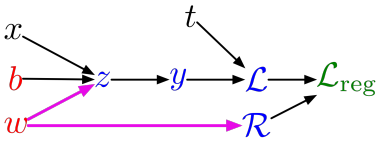
$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Values already computed by our program



In our new notation: $\overline{t} = \overline{x}\frac{\mathrm{d}x}{\mathrm{d}t} + \overline{y}\frac{\mathrm{d}y}{\mathrm{d}t}$

Backpropagation for Regularized Logistic Least Squares



Forward pass:

$$z = wx + b$$
$$y = \sigma(z)$$
$$\mathcal{L} = \frac{1}{2}(y - t)^2$$
$$\mathcal{R} = \frac{1}{2}w^2$$
$$\mathcal{L}_{\text{reg}} = \mathcal{L} + \lambda\mathcal{R}$$

Backward pass:

$$\overline{\mathcal{L}_{\text{reg}}} = 1$$

$$\overline{\mathcal{R}} = \frac{\mathrm{d}\mathcal{L}_{\text{reg}}}{\mathrm{d}\mathcal{R}} = \lambda$$

$$\overline{\mathcal{L}} = \frac{\mathrm{d}\mathcal{L}_{\text{reg}}}{\mathrm{d}\mathcal{L}} = 1$$

$$\overline{y} = \overline{\mathcal{L}}\,\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}y} = \overline{\mathcal{L}}\,(y - t)$$

$$\overline{z} = \overline{y}\,\frac{\mathrm{d}y}{\mathrm{d}z} = \overline{y}\,\sigma'(z)$$

$$\overline{w} = \overline{z}\,\frac{\partial z}{\partial w} + \overline{\mathcal{R}}\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}w} = \overline{z}\,x + \overline{\mathcal{R}}\,w$$

$$\overline{b} = \overline{z}\,\frac{\partial z}{\partial b} = \overline{z}$$

Full Backpropagation Algorithm:

Let $v_1, \ldots, v_N$ be an ordering of the computation graph where parents come before children (aka topological ordering).

v_N denotes the variable for which we try to compute gradients ($\mathcal{L}$, $\mathcal{L}_{\text{reg}}$ etc).

- forward pass:

$$\text{For } i = 1, \ldots, N,$$
$$\text{Compute } v_i \text{ as a function of Parents}(v_i).$$

- backward pass:

$$\bar{v}_N = 1$$
$$\text{For } i = N-1, \ldots, 1,$$
$$\bar{v}_i = \sum_{j \in \text{Children}(v_i)} \bar{v}_j \frac{\partial v_j}{\partial v_i}$$

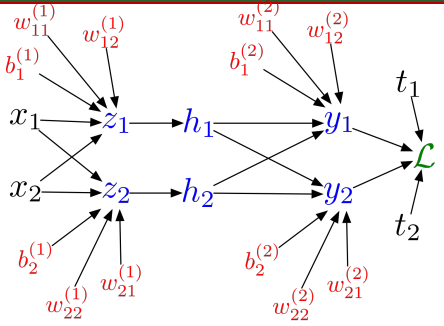
Backpropagation

- The algorithm for efficiently computing gradients in neural nets.
- Gradient descent with gradients computed via backprop is used to train the overwhelming majority of neural nets today.
- Even optimization algorithms much fancier than gradient descent (e.g. second-order methods) use backprop to compute the gradients.

Auto-Differentiation

- Autodifferentiation performs backprop in a completely mechanical and automatic way.
- Many autodiff libraries: PyTorch, Tensorflow, Jax, etc.
- Although autodiff automates the backward pass for you, it's still important to know how things work under the hood.
- In the tutorial, we will use an autodiff framework to build complex neural networks.

Backpropagation for Two-Layer Neural Network



Forward pass:

$$z_i = \sum_j w_{ij}^{(1)} x_j + b_i^{(1)}, \quad h_i = \sigma(z_i)$$

$$y_k = \sum_i w_{ki}^{(2)} h_i + b_k^{(2)}, \quad \mathcal{L} = \frac{1}{2} \sum_k (y_k - t_k)^2$$

Backward pass:

$$\overline{\mathcal{L}} = 1$$

$$\overline{y_k} = \overline{\mathcal{L}} \, (y_k - t_k)$$

$$\overline{w_{ki}^{(2)}} = \overline{y_k} \, h_i$$

$$\overline{b_k^{(2)}} = \overline{y_k}$$

$$\overline{h_i} = \sum_k \overline{y_k} w_{ki}^{(2)}$$

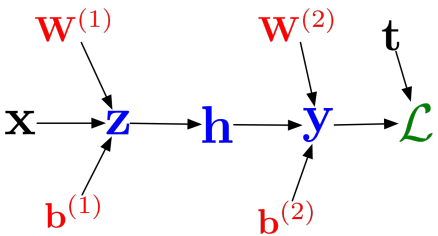
$$\overline{z_i} = \overline{h_i} \, \sigma'(z_i)$$

$$\overline{w_{ij}^{(1)}} = \overline{z_i} \, x_j$$

$$\overline{b_i^{(1)}} = \overline{z_i}$$

Backpropagation for Two-Layer Neural Network

In vectorized form:



Forward pass:

$$\mathbf{z} = \mathbf{W}^{(1)}\mathbf{x} + \mathbf{b}^{(1)}, \quad \mathbf{h} = \sigma(\mathbf{z})$$

$$\mathbf{y} = \mathbf{W}^{(2)}\mathbf{h} + \mathbf{b}^{(2)}, \quad \mathcal{L} = \frac{1}{2}\|\mathbf{t} - \mathbf{y}\|^2$$

Backward pass:

$$\overline{\mathcal{L}} = 1$$

$$\overline{\mathbf{y}} = \overline{\mathcal{L}}(\mathbf{y} - \mathbf{t})$$

$$\overline{\mathbf{W}^{(2)}} = \overline{\mathbf{y}}\mathbf{h}^\top$$

$$\overline{\mathbf{b}^{(2)}} = \overline{\mathbf{y}}$$

$$\overline{\mathbf{h}} = \mathbf{W}^{(2)\top}\overline{\mathbf{y}}$$

$$\overline{\mathbf{z}} = \overline{\mathbf{h}} \circ \sigma'(\mathbf{z})$$

$$\overline{\mathbf{W}^{(1)}} = \overline{\mathbf{z}}\mathbf{x}^\top$$

$$\overline{\mathbf{b}^{(1)}} = \overline{\mathbf{z}}$$

Conclusion

Today we discussed neural networks:

- We discussed their expressive power.
  - Can approximate any function (roughly speaking).
- Introduced backpropagation.
  - We also worked out the updates for a two-layer neural network.
- Please fill out course evaluations!