Advanced theory of statistics – Homework 1, Winter 2024/2025.

Please upload your solution to Quercus by February 3.

Problem 1. [Log-Likelihood in Curved Exponential Families] Let $f(x;\theta)$ represent an exponential family with log-partition function A and canonical parameter $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^d$. Consider a smooth and injective map $g : \mathbb{R}^k \to \mathbb{R}^d$ (for $k < d$) and suppose we constrain $\boldsymbol{\theta}$ to the subset

$$\Theta_0 = \{\boldsymbol{\theta} \in \Theta : \ \boldsymbol{\theta} = g(\boldsymbol{\eta}) \text{ for some } \boldsymbol{\eta} \in \mathbb{R}^k\}.$$

This creates a curved exponential family with parameter $\boldsymbol{\eta}$.

1. Derive the log-likelihood function for this curved family based on i.i.d. observations $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$.
2. Show that if $g(\boldsymbol{\eta})$ is linear, the resulting log-likelihood is concave in $\boldsymbol{\eta}$.
3. Provide a counterexample where $g(\boldsymbol{\eta})$ is nonlinear, and show that the log-likelihood is not guaranteed to be concave in $\boldsymbol{\eta}$.
4. Discuss how this lack of concavity can be dealt with in practice. How to optimize this function, how to verify that we found a local optimum?

Problem 2. Consider a minimal exponential family with density

$$f(x;\theta) = h(x) \exp\left\{\langle \theta, t(x)\rangle - A(\theta)\right\}, \quad \theta \in \Theta.$$

1. Suppose that $t(x)$ is replaced by a new sufficient statistic $s(x)$ such that $s(x) = Mt(x)$ for a fixed invertible matrix $M \in \mathbb{R}^{d\times d}$. Write the new density function in terms of $s(x)$ and the transformed parameter.
2. Prove that $s(x)$ remains a sufficient statistic, and discuss whether the resulting exponential family is minimal.
3. Construct a specific example of a transformation M for the bivariate Gaussian distribution (with both mean and variance unknown) and derive the transformed parameters explicitly.

Problem 3. Consider the exponential random graph model, $\mathbf{x} = (x_{ij}) \in \{0,1\}^{\binom{n}{2}}$ with $n \geq 3$, with $(n+1)$-dimensional sufficient statistics given by the number of edges of the graph and the degrees of each of the n nodes.

(i) Describe the space of the canonical parameters and write the formula for $t(\mathbf{x})$.

(ii) Is it a minimal/regular exponential family?

(iii) Find the formula for the maximum likelihood estimator expressed in terms of the mean parameters. Note that obtaining the MLE for the canonical parameter is hard.

(iv) Compote the log-partition function.