Advanced techniques in applied economics

Lecture 7: Positive dependence and total positivity

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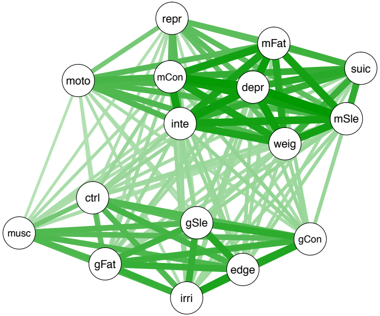
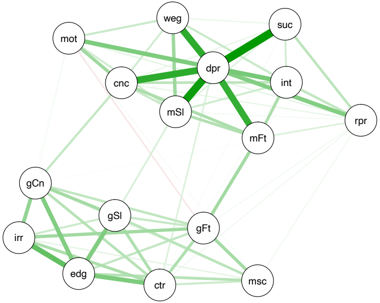
Motivation

Three motivating pictures

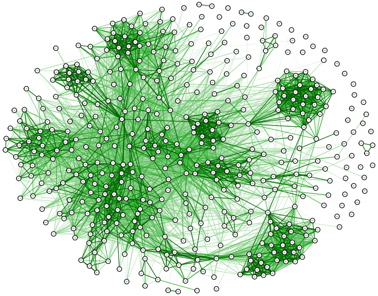
Depression/anxiety data



Ising fit



Stocks



In all three examples, most visible links are positive.

Today the question is: when can positive dependence itself be treated as useful structure?

Why talk about positive dependence?

Most of the course focused on sparsity, conditional independence, latent structure, and causality. Today we look at a different kind of structural assumption: the **sign** of dependence.

Economic examples

- stock indices often move together,
- sectoral returns react to common shocks,
- macro indicators comove over the business cycle,
- regional housing prices may rise together,
- bank stress measures often rise together in bad times.

Statistical payoff Sign constraints can act as regularization:

- estimation becomes easier,
- fitted precision matrices become sparse,
- covariance estimation becomes more stable,
- models can remain useful in higher dimensions.

Main message: knowing that dependence is positive is already a structural assumption — and sometimes a very useful one.

What positive dependence can mean in economics

Common-shock story A hidden force pushes many variables in the same direction: market return, business-cycle conditions, liquidity stress, local demand, inflation pressure, or sovereign risk.

What we gain statistically
If positive dependence is plausible, then we should build it into the model rather than estimate an unrestricted covariance matrix and hope for the best.

This lecture is about turning that qualitative idea into a formal modeling constraint.

Examples

- equity returns under a common market factor,
- employment across sectors in an expansion,
- house prices across nearby regions,
- sovereign spreads during a crisis.

Roadmap for today

1. positive association and MTP_2
2. Gaussian MTP_2 and the precision matrix
3. why this matters statistically
4. finance and factor-model examples
5. sign switching and signed MTP_2
6. beyond undirected models: positivity in linear Gaussian SEMs

Sparsity is not the only useful structure in dependence modeling.

The sign of dependence can be informative too.

Part 1: Association and MTP_2

Positive association

A random vector $X = (X_1, \ldots, X_m)$ is **positively associated** if for any two non-decreasing functions ϕ, ψ,

$$\mathrm{Cov}\{\phi(X), \psi(X)\} \geq 0.$$

Interpretation If one part of the system is high, then other increasing summaries also tend to be high. So this goes beyond pairwise correlations: it is a statement about the whole joint distribution.

Gaussian fact [Pitt] A Gaussian vector is positively associated if and only if all entries of Σ are nonnegative.

Economic examples

- multiple stock returns,
- industrial production across sectors,
- survey answers measuring confidence,
- regional activity indicators.

For classical foundations of positive dependence, see Karlin and Rinott (1980).
For the Gaussian equivalence, see Pitt (1982).

Multivariate total positivity of order two

A density f is MTP_2 if for all x, y,

$$f(x)f(y) \leq f(x \wedge y)\, f(x \vee y),$$

where $x \wedge y$ and $x \vee y$ are the componentwise minimum and maximum.

Why this matters MTP_2 is a strong and tractable form of positive dependence. It implies positive association and behaves well under many basic operations.

If X is MTP_2, then marginals and conditionals are again MTP_2.

$\mathrm{MTP}_2 \implies$ Positive Association

Smooth case

If $\log f$ is twice differentiable, then MTP_2 means that all mixed second derivatives of $\log f$ are nonnegative.

For MTP_2 and its structural properties, see Karlin and Rinott (1980), Fallat et al. (2017), and Lauritzen, Uhler, and Zwiernik (2019).

A binary example: EPH-gestosis

- EPH-gestosis is a pregnancy syndrome involving three symptoms: high body water retention, protein in urine, and elevated blood pressure.
- A syndrome is exactly the kind of setting where one expects positive dependence.

The empirical distribution

$$\begin{bmatrix} \hat{p}_{000} & \hat{p}_{010} & \hat{p}_{001} & \hat{p}_{011} \\ \hat{p}_{100} & \hat{p}_{110} & \hat{p}_{101} & \hat{p}_{111} \end{bmatrix} = \frac{1}{4649} \begin{bmatrix} 3299 & 107 & 1012 & 58 \\ 78 & 11 & 65 & 19 \end{bmatrix}$$

is already MTP_2.

Interpretation

The observed dependence is compatible with a simple latent binary explanation:

$$X_1 \perp\!\!\!\perp X_2 \perp\!\!\!\perp X_3 \mid H$$

for a hidden binary variable H, with all three symptoms positively related to H.

Part 2: Gaussian MTP_2

Gaussian MTP_2

Suppose

$$X \sim N(\mu, \Sigma), \qquad K = \Sigma^{-1}.$$

Gaussian characterization

The Gaussian distribution is MTP_2 if and only if

$$K_{ij} \leq 0 \qquad \text{for all } i \neq j.$$

So Gaussian total positivity is exactly a sign constraint on the precision matrix.

Consequences

- all marginal correlations are nonnegative,
- all partial correlations are nonnegative,

This is much stronger than merely asking for positive pairwise correlations. It constrains the full conditional dependence structure.

For the Gaussian characterization, see Lauritzen, Uhler, and Zwiernik (2019).

A binary analogue

Positive dependence also appears naturally in binary models.

Ising model

$$p(x) \propto \exp\big(h^\top x + x^\top Jx/2\big).$$

If

$$J_{ij} \geq 0 \qquad (i \neq j),$$

then the model is attractive.

Why mention this?

Relevant in survey data, psychometrics, and symptom networks.

For MTP_2 in exponential families, including binary models, see Lauritzen, Uhler, and Zwiernik (2021).

Is MTP_2 too restrictive?

Without structure
Among arbitrary Gaussian distributions, MTP_2 is indeed restrictive. So it is not a generic default model:

- 3D Gaussian: about 5% of distributions are MTP_2,
- 4D Gaussian: about 0.1% of distributions are MTP_2.
- In higher dimensions the probability of sampling a covariance matrix consistent with the MTP_2 is essentially zero.

With structure
Once sparsity, trees, latent factors, or one-sided economic mechanisms are already plausible, MTP_2 becomes much less exotic.

Key intuition

Positive dependence is especially natural when many variables respond to the same hidden force in the same direction.

So the right question is not *"is MTP_2 common among arbitrary covariance matrices?"* but *"is it plausible in the structured model class I care about?"*

A financial example

Monthly correlations among major stock-market indices are all positive:

$$S = \begin{pmatrix} 1 & 0.606 & 0.731 & 0.618 & 0.613 \\ 0.606 & 1 & 0.550 & 0.661 & 0.598 \\ 0.731 & 0.550 & 1 & 0.644 & 0.569 \\ 0.618 & 0.661 & 0.644 & 1 & 0.615 \\ 0.613 & 0.598 & 0.569 & 0.615 & 1 \end{pmatrix}$$

$$S^{-1} \approx \begin{pmatrix} 2.629 & -0.480 & -1.249 & -0.202 & -0.490 \\ -0.480 & 2.109 & -0.039 & -0.790 & -0.459 \\ -1.249 & -0.039 & 2.491 & -0.675 & -0.213 \\ -0.202 & -0.790 & -0.675 & 2.378 & -0.482 \\ -0.490 & -0.459 & -0.213 & -0.482 & 1.992 \end{pmatrix}.$$

This sample is already essentially Gaussian MTP_2.

Part 3: Why MTP_2 is statistically useful?

Why economists should care

The sign restriction is not just philosophically appealing. It improves the statistics.

Key facts in the Gaussian case

- the MLE under MTP_2 exists already with very small samples,
- the fitted inverse covariance is automatically sparse,
- the sign constraint acts like built-in regularization.

Practical message

This is useful in finance and macro, where positive dependence is often plausible.

For existence, sparsity, and MLE structure, see Slawski and Hein (2015) and Lauritzen, Uhler, and Zwiernik (2019).

The estimation problem

Given a sample covariance matrix S, we estimate a Gaussian MTP_2 model by solving

$$\min_{K \succ 0} \left\{ -\log\det K + \operatorname{tr}(SK) \right\}$$

subject to

$$K_{ij} \leq 0 \qquad (i \neq j).$$

Interpretation This is a Gaussian log-likelihood under a sign constraint on the precision matrix.

The sign restriction replaces part of the usual structural prior knowledge.

So MTP_2 is one example where economically meaningful structure also leads to computational and statistical simplification.

Recent computational directions

Recent work pushes Gaussian MTP_2 estimation in directions that matter for practice.

Examples

- tuning-free structure learning in high dimension,
- adaptive estimation of sign-constrained precision matrices,
- bridge-block decompositions for large-scale Gaussian MTP_2 models.

Practical lesson

The modern literature is not only about theory. A lot of it is about making positive-dependence models scalable and usable.

See Wang, Roy, and Uhler (2020), Ying, Cardoso, and Palomar (2023), and Wang, Ying, and Palomar (2023).

Part 4: Factor-model and finance examples

Why positive dependence is natural under common shocks

One-factor mechanism If $X_i = \lambda_i F + \varepsilon_i$, with $\lambda_i > 0$, then

$$\mathrm{Cov}(X_i, X_j) = \lambda_i \lambda_j \,\mathrm{Var}(F) \geq 0.$$

So one common factor with positive loadings automatically produces positive dependence.

Precision-matrix: Algebraically

$$\Sigma = \tau^2 \lambda \lambda^\top + \Psi, \qquad \Psi \text{ diagonal.}$$

Using Woodbury formula:

$$(\Sigma^{-1})_{ij} = -\frac{\tau^2 \lambda_i \lambda_j \psi_i^{-1} \psi_j^{-1}}{1+\tau^2 \lambda^\top \Psi^{-1} \lambda} < 0, \quad i \neq j.$$

Economic examples

- stock returns and the market factor,
- sectoral output and the business cycle,
- regional house prices and local demand,

Relevance

This is why positive dependence is not an exotic mathematical idea. It is often the direct consequence of a familiar economic story.

Why this matters for portfolios

In the classical Markowitz setup, the **global minimum-variance portfolio** solves

$$\min_{w} \; w^\top \Sigma w \qquad \text{subject to} \qquad \mathbf{1}^\top w = 1.$$

Closed-form solution

The optimizer is $w^\star = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}}$. In practice, we replace the unknown Σ by an estimator $\hat{\Sigma}$, giving

$$\hat{w} = \frac{\hat{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top \hat{\Sigma}^{-1}\mathbf{1}}.$$

Where the difficulty is
The weights depend on the *inverse covariance matrix*, not just on $\hat{\Sigma}$ itself. Small estimation errors in $\hat{\Sigma}$ can become much larger after inversion.

Particularly bad in high dimensions.

In this application the MTP_2 assumption is particularly powerful.

This also regularizes the inverse covariance matrix and can produce more stable portfolio weights.

For this application of MTP_2, see Agrawal, Roy, and Uhler (2024), *The Journal of Finance*.

What the portfolio paper compares

The paper compares covariance estimators for portfolio construction.

Competing approaches

- sample covariance,
- linear and nonlinear shrinkage,
- factor-model estimators such as POET,
- Gaussian MTP_2 estimators,
- robust Kendall's tau based MTP_2 estimators.

Main question

Can a positive-dependence constraint give better out-of-sample portfolios than more standard regularization methods?

Main outcomes from the portfolio paper

Global minimum-variance setting
Across the simulation settings, the MTP_2 estimators were consistently among the strongest methods for out-of-sample risk, and often clearly improved over the raw sample covariance.

Mean-variance setting
When expected returns were incorporated through a signal, the robust Kendall-based MTP_2 version was especially competitive, often outperforming shrinkage and factor-based alternatives.

Take-away

A simple sign constraint can be competitive with, and sometimes better than, much more standard covariance estimators in portfolio problems.

A Gaussian MTP_2 fit in R

A package route is to use golazo for sign-constrained Gaussian graphical models.

```
# if needed:
# devtools::install_github("pzwiernik/golazo")

library(golazo)
R <- cor(dat)
res <- positive.golazo(R, rho = Inf)
K <- res$K
```

```
library(qgraph)
qgraph(-cov2cor(K),
       layout = "spring",
       threshold = 0.05)
```

Interpretation

The estimated precision matrix respects the positive-dependence sign constraints and often becomes sparse automatically.

A robust variation

For heavy-tailed financial data, one can combine positive dependence with rank-based correlations.

```
kR <- cor(dat, method = "kendall")
sR <- sin(pi * kR / 2)
res <- positive.golazo(sR, rho = Inf)
K <- res$K
```

```
qgraph(-cov2cor(K),
       layout = "spring")
```

Interpretation

This combines the rank-based nonparanormal idea with the positive-dependence constraint.

Part 5: Positivity after switching signs

Sometimes positivity appears after changing signs

Not every meaningful dataset is positively dependent in its original coordinates.

A common hidden force may still be present, but some variables are coded in the opposite direction.

Sometimes the right question is not *"is the data positively dependent?"* but *"can the data be made positively dependent after a sensible reorientation of signs?"*

Economic examples

- unemployment instead of employment,
- credit spread instead of credit quality,
- losses instead of gains,
- inflation instead of purchasing power,
- distress indices instead of prosperity indices.

Signed MTP_2 idea

Sometimes there exists a diagonal sign matrix

$$D = \mathrm{diag}(\pm 1, \ldots, \pm 1)$$

such that DX is MTP_2.

Part 6: Positivity in linear Gaussian SEMs

A directed analogue: positivity in linear Gaussian SEMs

So far positivity was discussed mainly through covariance and undirected graphical structure.

A different question is: what does positive dependence look like in Gaussian DAG models?

Setup

In a Gaussian SEM $X = BX + \varepsilon$, one may require the structural coefficients to be nonnegative.

Main message

This gives a directed notion of positivity: parents push children in the same direction. Recent work shows that this is closely tied to the CIS property and leads to useful estimation theory.

Monotonic effects. Some causal analysis literature focuses on whether increasing one variable can only increase or decrease another.

Difference That literature is mostly nonparametric and qualitative. Our focus is specifically on *linear Gaussian SEMs* and sign constraints on structural coefficients.

For monotonic effects and signed DAGs, see VanderWeele and Robins (2009). For the Gaussian SEM perspective, see Lodhia, Hütter, Uhler, and Zwiernik (2026) .

Take-away

- Positive dependence is a structural assumption, not only a descriptive comment.
- MTP_2 is a strong and useful formalization of this idea.
- In Gaussian models, MTP_2 is equivalent to a sign constraint on the precision matrix.
- This sign structure can improve covariance and graph estimation in practice.
- Positive dependence often appears naturally in finance, factor models, and sometimes only after sensible sign changes.
- There is also a directed analogue through positivity in linear Gaussian SEMs.

Sparsity is not the only useful structure in dependence modeling. The sign of dependence can be informative too.

Final course perspective

Across the course, useful models were those that encode the right kind of structure

- independence and sparsity,
- direction and interventions,
- latent variables and hidden structure,
- hidden confounding and partial adjustment,
- and now the sign of dependence.

Final thought

A useful model is often one that encodes the right structural simplification: independence, sparsity, latentness, direction, confounding structure, or sign.