Universitat Pompeu Fabra Barcelona

Lecture 8 · Random Networks II

Networks, Crowds and Markets

Today's Lecture

1. Probability recap: Chebyshev and Hoeffding inequality.
2. Degree distribution in Erdős–Rényi graphs.
3. Threshold phenomena and giant component.
4. The clustering coefficient: definition, motivation, formulas.
5. Static random graph models: ER as binary vectors, ERGMs.
6. Recursive random graph models: preferential attachment.
7. Why random models matter for economics and social sciences.

Degree distribution: finite N concentration bounds

Concentration: Chebyshev (simple but general)

Theorem (Chebyshev inequality)

For any r.v. X with mean μ and variance σ^2,

$$\mathbb{P}(|X-\mu| \geq t) \ \leq \ \frac{\sigma^2}{t^2}.$$

For degree: $\deg(v) \sim \mathrm{Bin}(N-1, p)$, so

$$\mathbb{P}\big(|\deg(v) - (N-1)p| \geq t\big) \leq \frac{(N-1)p(1-p)}{t^2}.$$

Chebyshev already gives some concentration guarantees (e.g. take $t_0 = \sqrt{\frac{N}{\delta}p(1-p)}$ for small $\delta > 0$) but sharper results are possible.

Appendix: Proof of the Chebyshev inequality

Markov's inequality: If $Z \geq 0$ then $\mathbb{P}(Z \geq t) \leq \frac{1}{t}\mathbb{E}[Z]$.

Markov's inequality follows immediately from the following calculation,

$$\mathbb{E}[Z] \;\leq\; \mathbb{E}[Z\mathbb{1}(Z \geq t)] \;\leq\; t\mathbb{E}[\mathbb{1}(Z \geq t)] \;=\; t\mathbb{P}(Z \geq t).$$

Now, Chebyshev's inequality follows easily from Markov's. Take $Z = |X - \mu|$ then

$$\mathbb{P}(|X-\mu| \geq t) \;=\; \mathbb{P}((X-\mu)^2 \geq t^2) \;\leq\; \frac{\mathbb{E}(X-\mu)^2}{t^2} \;=\; \frac{\sigma^2}{t^2}.$$

Sharper concentration: Hoeffding for Binomial

Theorem (Hoeffding inequality)

If $X = \sum_{i=1}^{n} Z_i$ with independent $Z_i \in [0,1]$ and $\mathbb{E}X = \mu$, then for $t > 0$,

$$\mathbb{P}(|X - \mu| \geq t) \quad \leq \quad 2\exp\Big(-\frac{2t^2}{n}\Big).$$

Applied to degree: $\deg(v)$ has $N-1$ independent Bernoulli summands,

$$\mathbb{P}(|\deg(v) - (N-1)p| \geq t) \ \leq \ 2\exp\left(-\frac{2t^2}{N-1}\right).$$

Fix $v \in V$. Taking $t_0 = \sqrt{\frac{N-1}{2}\log(\frac{2}{\delta})}$ for small $\delta > 0$ gives

$$\mathbb{P}\big(|\deg(v) - (N-1)p| \geq t_0\big) \ \leq \ \delta.$$

note much better behavior of t_0 as a function of δ

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e.g. $N = 1001$, $p = 0.1$, $\delta = 0.05$. Then with prob. ≥ 0.95

$\deg(v) \;\in\; (100-42.95, 100+42.95) \;=\; (57.05, 142.95).$

Uniform degree bounds

Recall: $\mathbb{P}(|\deg(v) - (N-1)p| \geq t) \leq 2\exp\left(-\frac{2t^2}{N-1}\right)$ for all $t > 0$.

Suppose we now want to provide a bound for the degrees all $v \in V$.

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Union bound: For any two events $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$.

$$\mathbb{P}(\exists v \;\; |\deg(v) - (N-1)p| \geq t_0) \;\leq\; \sum_{v \in V} \mathbb{P}(|\deg(v) - (N-1)p| \geq t_0) \;\leq\; \delta.$$

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e.g. $N = 1001$, $\delta = 0.05$, $p = 0.1$. Then with prob. ≥ 0.95 all degrees lie in $(100 - 72.8, 100 + 72.8) = (27.2, 172.8)$.

Asymptotics in networks

Asymptotic Thinking in Random Graphs

Why asymptotics?

- We study $G(N, p)$ as $N \to \infty$ to reveal general patterns.
- Precise constants matter less than the scaling behavior of p with N.

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- $f(N) = O(g(N))$ means $|f(N)| \leq C|g(N)|$; for some $C > 0$ and N large enough.
- $f(N) \sim g(N)$ means $f(N)/g(N) \to 1$.

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Probabilistic language:

- "With high probability" (w.h.p.) means $\mathbb{P}(\text{event}) \to 1$ as $N \to \infty$.
- Example: in $G(N, p)$ with $p = \frac{\log N}{N}$, the graph is connected *w.h.p.*

Average degree: dense vs sparse graphs

When N grows, the connection probability $p = p_N$ can scale differently.

Dense regime: (p_N) tends to a constant $c > 0$.

- $\mathbb{E}[\deg(v)] \approx cN$ grows linearly with N.
- The number of edges $L \approx c\binom{N}{2}$.
- Not a realistic large network, but a useful contrast.

Sparse regime: $p_N = \lambda/N$ (or smaller).

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- The total number of edges $L \approx \lambda N/2$ grows linearly with N.

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Language note:

- Saying "real networks are sparse" means that as they grow, the *average degree* stays bounded, not that p is small for a fixed N.
- The scaling of p_N determines which asymptotic regime we are in.

Maximum degree in $G(N, p)$

Let $\Delta = \max_v \deg(v)$ be the **maximum degree**.

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- With high probability (remember we ignore constants here):

$$\Delta \;=\; (N-1)p \,+\, O\big(\sqrt{N \log N}\big).$$

(use Slide 6 to argue for this asymptotic formula)

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Sparse regime: $p_N = \lambda/N$ (or smaller).

- Each $\deg(v) \approx \mathrm{Pois}(\lambda)$ — mean λ.
- By extreme–value theory for Poisson tails:

$$\Delta \;\approx\; \frac{\log N}{\log\log N}.$$

This is very thin tailed: $N = 10^3, 10^6, 10^{12}$ gives $\frac{\log N}{\log\log N} = 4.3, 6.3, 9.2.$
In real networks we observe "hubs".

Threshold phenomena and giant component

Threshold phenomena in ER (concept)

Definition

A **threshold** for a graph property $\mathcal{P}$ is a function $p^*(N)$ such that:

$$p \ll p^*(N) \Rightarrow G(N,p) \text{ has } \neg\mathcal{P} \text{ w.h.p.},$$

$$p \gg p^*(N) \Rightarrow G(N,p) \text{ has } \mathcal{P} \text{ w.h.p.}$$

ER graphs display many sharp thresholds:

- Emergence of a giant component.
- Connectivity (no isolated vertices).
- Appearance of fixed subgraphs (e.g., triangles).

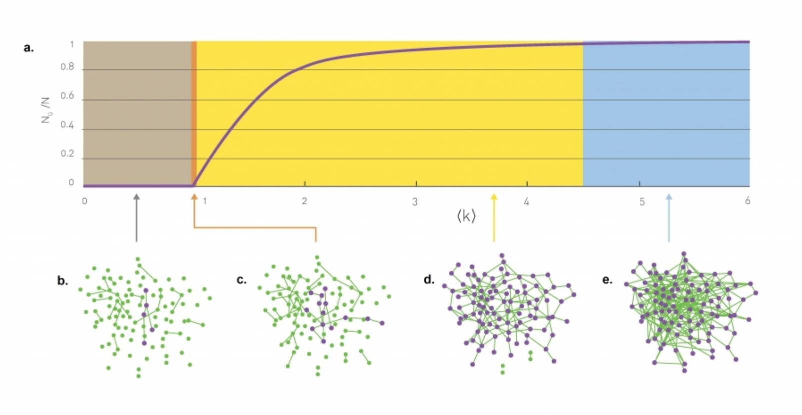
Regimes of $G(N, p)$ (sparse case $p = c/N$)

It is useful to describe random graphs in terms of the **expected degree**

$$\mathbb{E}[\deg(v)] \;=\; c.$$

- **Subcritical regime ($c < 1$):** only small tree-like components; largest size $\sim \log N$.
- **Critical point ($c = 1$):** largest component has size $\sim N^{2/3}$; no giant yet.
- **Supercritical regime ($c > 1$):** a unique **giant component** emerges, containing a positive fraction of nodes.
- **Connected regime ($c \gtrsim \log N$):** almost surely the whole graph becomes connected.

Illustration of regimes



Interpretation: As c increases, the largest connected component grows from negligible size, through a sudden phase transition ($c = 1$), and eventually absorbs almost all nodes.

Why the giant component matters (econ/social)

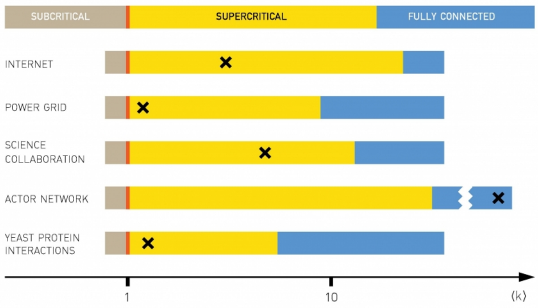
Consider the world's friendship network:

- Clearly disconnected (think small remote communities)
- But "our" component is large, spans most of the world.
- There should be no two big components.

Giant components are important:

- **Contagion & diffusion:** A giant component enables large cascades (diseases, information, bank runs).
- **Market connectivity:** Sufficient density is needed for trade/payment networks to connect most participants.
- **Infrastructure design:** Tuning p (or expected degree c) above 1 ensures large-scale reachability.

Where are real networks?



Most real-world networks live **well above the critical point**.

They are highly connected (often even "superconnected"), yet they also exhibit additional structure (clustering, hubs, communities).

The ER model a *baseline*: it shows that above $c = 1$, large-scale connectivity is the default, but real networks have richer features.

Connectivity threshold in $G(N, p)$

Theorem

The threshold for connectivity in $G(N, p)$ is

$$p^*(N) = \frac{\log N}{N}.$$

More precisely:

$$\begin{cases} p = \frac{\log N + \omega(N)}{N}, & G(N, p) \text{ is connected w.h.p.,} \\ p = \frac{\log N - \omega(N)}{N}, & G(N, p) \text{ is disconnected w.h.p..} \end{cases}$$

Here, $\omega(N)$ means any function that grows to infinity (however slowly).
Examples: $\log \log N$, $\sqrt{\log N}$, or even $\log \log \log N$.

Idea of proof (intuition)

- A vertex is isolated with probability

$$\mathbb{P}(v \text{ isolated}) \;=\; (1-p)^{N-1} \;\approx\; e^{-pN}.$$

- Expected number of isolated vertices:

$$\mathbb{E}[N_0] \approx Ne^{-pN}.$$

- If $p = c\frac{\log N}{N}$, then

$$\mathbb{E}[N_0] \approx N^{1-c}.$$

- For $c < 1$, $\mathbb{E}[N_0] \to \infty$; many isolated vertices $\to$ disconnected.

  For $c > 1$, $\mathbb{E}[N_0] \to 0$; isolated vertices disappear.

Careful: No isolated vertices do not automatically imply connectivity. However, one can show that once all isolated vertices disappear, all other components merge into one giant component w.h.p.

Simulation in NetworkX (Colab) — generate and inspect

Python (run in Google Colab)

```
import networkx as nx
import matplotlib.pyplot as plt

n, p = 200, 0.015  # try also p = 0.005, 0.02, 0.05
G = nx.erdos_renyi_graph(n, p)

print("Nodes:", G.number_of_nodes())
print("Edges:", G.number_of_edges())

# Empirical vs expected average degree
deg = [d for _, d in G.degree()]
print("Empirical mean degree:", sum(deg)/n)
print("Theoretical mean degree:", (N-1)*p)

# Largest component size
components = list(nx.connected_components(G))
largest = max(components, key=len)
print("Largest component size:", len(largest))

# Draw (small n looks better)
plt.figure(figsize=(5,5))
pos = nx.spring_layout(G, seed=7)
nx.draw(G, pos, node_size=30, edge_color="#cccccc")
plt.show()
```

Simulation in NetworkX — degree histogram

Python (run in Google Colab)

```
import numpy as np
import matplotlib.pyplot as plt

deg = np.array([d for _, d in G.degree()])
print("Empirical mean degree:", deg.mean())
print("Theoretical mean degree:", (N-1)*p)

plt.figure(figsize=(5,4))
bins = np.arange(deg.max()+2) - 0.5
plt.hist(deg, bins=bins)
plt.xlabel("Degree k"); plt.ylabel("Count")
plt.title("Degree distribution in G(N,p)")
plt.show()
```

Observation. For $p = c/N$ the histogram should resemble a Poisson(c), with empirical mean degree $\overline{\mathrm{deg}}(G)$ close to theoretical $\mathbb{E}[\mathrm{deg}]$.

Clustering

Why clustering matters

Real networks are not tree-like. Friends of friends often know each other (and so triangles are common).

Examples:

- **Social networks:** If Alice knows Bob and Carol, it's likely Bob and Carol also know each other. $\rightarrow$ Social circles, community structure.
- **Trade networks:** Countries trading with the same partner often trade with each other. $\rightarrow$ Formation of regional trade blocs.
- **Financial networks:** Two banks lending to the same counterparties are likely connected through risk exposures. $\rightarrow$ Triangles increase contagion channels.
- **Citation or collaboration networks:** If researcher A collaborates with both B and C, B–C collaboration becomes more probable. $\rightarrow$ Knowledge diffusion through closed triads.

Clustering coefficient: definition

Definition

For node v with degree $\deg(v) = k_v$:

$$C_v = \frac{\#\text{ links among neighbors of } v}{\binom{k_v}{2}} \in [0,1].$$

- Measures "friend-of-friend closure."
- $C_v = 1$: neighbors form a clique; $C_v = 0$: none connected.
- Average clustering coefficient: $\overline{C} = \frac{1}{N}\sum_v C_v$.

Clustering in Erdős–Rényi networks

Suppose $\deg(v) = k_v$. Consider two neighbors u, w.

Each pair u, w gets connected (independently) with probability p.

The expected number of links among neighbors is $\mathbb{E}L_v = p\binom{k_v}{2}$.

Thus

$$\mathbb{E}[C_v] \;=\; \mathbb{E}\left[\frac{L_v}{\binom{k_v}{2}}\right] \;=\; \frac{\mathbb{E}[L_v]}{\binom{k_v}{2}} \;=\; p.$$

Implications:

- In the sparse regime $p = c/N$: $\mathbb{E}[C_i] \approx c/N \to 0$.
- Prediction: clustering vanishes as N grows.
- Real networks (social, financial, trade) exhibit far higher clustering. $\Rightarrow$ Mismatch: motivates richer models leading so sparse networks with nontrivial clustering coefficients.

Summary: What ER graphs teach us (and what they miss)

Erdős–Rényi: clean benchmark for randomness in networks.

- Degrees: Binomial $\to$ Poisson in sparse regime, sharply concentrated (Hoeffding).
- Sharp thresholds: giant component at $p \sim 1/N$, full connectivity at $p \sim (\log N)/N$.

Analytic power: every property can be studied precisely—gives language for thresholds, asymptotics, and "with high probability" results.

But realism is limited:

- Clustering $\mathbb{E}[C_v] = p \to 0$ as $N \to \infty$ (in the sparse regime).
- Degree distribution thin-tailed: no hubs or communities.
- Real social, financial, and web networks are way more structured.

This motivates a study of other random graph models.

Static random graph models

Graphs as random objects

Consider an undirected graph $G = (V, E)$.

Order all pairs of elements in V: $\{1,2\}, \{1,3\}, \ldots, \{N-1, N\}$.

Each graph is uniquely identified by a vector $\boldsymbol{y} = (y_{ij}) \in \{0,1\}^{\binom{N}{2}}$:

- $y_{ij} = 1$ if and only if $ij \in E$.

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- $y_{ij} = 1$ if and only if $ij \in E$.

In this sense, every distribution for a random binary vector in $\{0,1\}^{\binom{N}{2}}$ gives a distribution of a random graph with N nodes.

e.g. $(p_{000}, p_{001}, p_{010}, p_{011}, p_{100}, p_{101}, p_{110}, p_{111}) = (\frac{1}{2}, \frac{1}{14}, \frac{1}{14}, \frac{1}{14}, \frac{1}{14}, \frac{1}{14}, \frac{1}{14}, \frac{1}{14})$ gives a distribution over 3-node graphs.

Every family of distributions over $\{0,1\}^{\binom{N}{2}}$ gives a statistical model for random graphs with N nodes.

Erdős–Rényi model as an example

Recall: Every family of distributions over $\{0,1\}^{\binom{N}{2}}$ gives a statistical model for random graphs with N nodes.

Consider the distribution where, for $\boldsymbol{y} = (y_{ij}) \in \{0,1\}^{\binom{N}{2}}$

$$p(\boldsymbol{y}) \;=\; (1-p)^{1-y_{12}} p^{y_{12}} \cdots (1-p)^{1-y_{N-1,N}} p^{y_{N-1,N}} \;=\; (1-p)^{\binom{N}{2}-s} p^s,$$

where $s = \sum_{i<j} y_{ij}$ is the number of edges.

Note: We can write $p(\boldsymbol{y}) = (1-p)^{\binom{N}{2}} \left(\frac{p}{1-p}\right)^s$.

Quick recall: exponential families

Let $\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}^n$, $T : \mathbb{R}^n \to \mathbb{R}^d$, $\theta \in \mathbb{R}^d$.

Definition

A probability distribution on $\mathcal{X}$ is an *exponential family* if the pms/density takes the form

$$p_\theta(\boldsymbol{x}) = h(\boldsymbol{x}) \exp\left(\theta^T T(\boldsymbol{x}) - \psi(\theta)\right).$$

- $T(\boldsymbol{x})$ = sufficient statistics (counts of edges, triangles, ...).
- θ = natural parameter.
- $\psi(\theta)$ = log-partition function (ensures normalization).

Logistic regression, Ising models, multivariate Gaussian, and many other popular statistical models are exponential families.

Static random graph models

Definition (Exponential Random Graph Models (ERGMs):)

$$\mathbb{P}(G = g) \propto \exp\{\theta_1 \cdot \#\text{edges}(g) + \theta_2 \cdot \#\text{triangles}(g) + \cdots\}.$$

- The parameters: θ_1 tunes density, θ_2 tunes clustering, etc.

ER model is a special case of ERGM:

$$\mathbb{P}(G = g) \; = \; (1 - p)^{\binom{N}{2}} \left(\frac{p}{1 - p} \right)^s \propto \exp(\theta \cdot s),$$

where $\theta = \log \left(\frac{p}{1-p} \right)$

Dynamic random graph models

Recursive growth: preferential attachment

Networks often grow over time (new users, new connections).

Preferential attachment: New node attaches to existing node v with probability proportional to $\deg(v)$.

- "Rich get richer" $\rightarrow$ hubs emerge.

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- "Rich get richer" $\rightarrow$ hubs emerge.

Result: degree distribution follows a *power law*.

- Few very large hubs.
- Many low-degree nodes.
- Matches data: web, citation networks, finance.

Summary

- $G(N, p)$ = simplest random graph; tractable but unrealistic.
- Subgraph thresholds (triangles) show how clustering begins.
- Clustering coefficient: vanishes in ER, but high in real networks.
- Static (ERGMs) and recursive (preferential attachment) models add realism.
- Small-world phenomena + hubs: explain short distances and inequalities.

Exercise

Determine the Clustering Coefficient for nodes w and y.