Universitat Pompeu Fabra
Barcelona

Lecture 6 · Centrality measures II

Networks, Crowds and Markets

Today's Lecture

1. Linear Algebra, Random walks, and PageRank
2. Why random graphs? Motivation and Erdős–Rényi models.
3. Probability recap for $G(N, p)$:
   3.1 Binomial distribution (edges, degrees).
   3.2 Poisson approximation in the sparse regime.

Basic spectral theory

Why Linear Algebra for Networks?

- Adjacency matrix A_G: encodes all links of G.
- Degree vector: $A_G\mathbf{1} = (\deg(v_1), \ldots, \deg(v_N))$.
- Laplacian $L = D - A_G$: central in diffusion, clustering, spanning trees.
- Many network measures (centrality, random walks, PageRank) reduce to eigenvalue/eigenvector problems.

Note

Eigenvalues of A_G reveal secrets of G.

- Google built its empire on one eigenvector (PageRank).
- Spotify/Youtube recommenders use eigenvector-like ideas.
- In social networks, eigenvector centrality captures being "friends with important people."

Recall: Eigenvalues and Eigenvectors

Definition

Let $A \in \mathbb{R}^{n \times n}$ then $\boldsymbol{v} \neq \mathbf{0}$ is called an eigenvector of A if

$$A\mathbf{v} = \lambda \mathbf{v}$$

for some λ, called eigenvalue. Assume $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}^\top \boldsymbol{v}} = 1$.

If A has only real eigenvalues then it can be diagonalized: $\exists$ invertible P s.t.

$$A \; = \; P \Lambda P^{-1} \quad \text{with } \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

The columns of P are the eigenvectors of A.

Note

If A is diagnosable then $A^k = P \Lambda^k P^{-1}$, $\Lambda^k = \operatorname{diag}(\lambda_1^k, \ldots, \lambda_n^k)$.

Spectral theorem

Theorem

If A is symmetric (i.e. $A = A^\top$), all eigenvalues are real, and eigenvectors form an orthogonal basis.

A is diagonalizable and for some orthogonal matrix U (i.e. $U^\top U = I_n$):

$$A = U \Lambda U^\top .$$

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Note (Variational characterization of eigenvectors)

The eigenvectors are the saddle points of $\boldsymbol{x}^\top A \boldsymbol{x}$ subject to $\|\boldsymbol{x}\| = 1$:

- By KKT condition each optimum is a stationary point of

$$\text{Lagrangian} \;\; = \;\; \boldsymbol{x}^\top A \boldsymbol{x} - \lambda(\boldsymbol{x}^\top \boldsymbol{x} - 1).$$

- This gives $A\boldsymbol{x} = \lambda \boldsymbol{x}$. And for every such unit $\boldsymbol{x}$, $\boldsymbol{x}^\top A \boldsymbol{x} = \lambda$.

In particular, the maximal eigenvalue is $\lambda_{\max} = \max_{\|\boldsymbol{x}\|=1} \boldsymbol{x}^\top A \boldsymbol{x}$.

Eigenvalue centrality

Motivation

In degree centrality all neighbours are treated equally.

Now: a node is important if connected to other important nodes.

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- We try to define an importance measure x_v for $v \in V$ s.t.

$$x_v \propto \sum_{u \sim v} x_u.$$

In matrix form: there exists $\lambda > 0$ and a positive $\boldsymbol{x}$ s.t.

$$A_G \boldsymbol{x} = \lambda \boldsymbol{x}.$$

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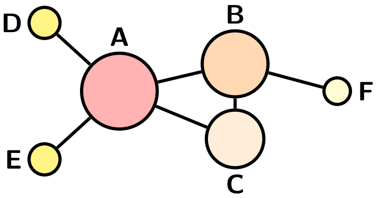
So centrality is given by an eigenvector of A_G with a positive eigenvalue.

Theorem (special case of Perron-Frobenius)

As A_G has nonnegative entries, maximal eigenvalue is positive.
Since $\mathbf{1}^\top A_G \mathbf{1} = 2L > 0$ then $\lambda_{\max} > 0$.
The principal eigenvector has positive entries.

Eigenvector Centrality – Core–Periphery Example



Adjacency matrix (A):

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Setup. A small core (A,B,C) connected as a triangle; three peripheral nodes (D,E,F) each attach to the core.

Why sizes differ.

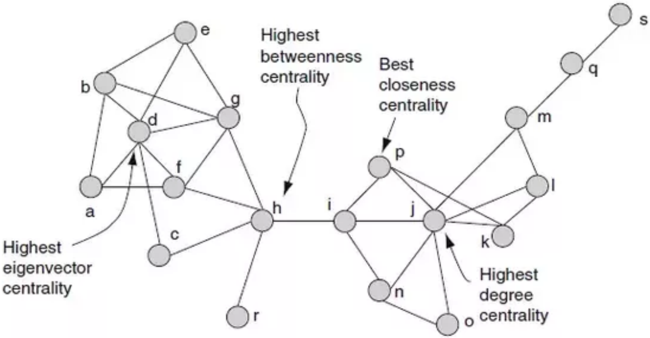
- ***A*** connects to two central nodes (***B***,***C***) and two peripherals (***D***,***E***) — very central.
- ***B*** beats ***C*** because it also connects to ***F***.
- ***D***, ***E***, ***F*** are peripheral and get low scores.

Note (Potential problems)

- What if G is disconnected?
- What if $\lambda_{\max}$ has multiplicity ≥ 2?

Normalized ratios:

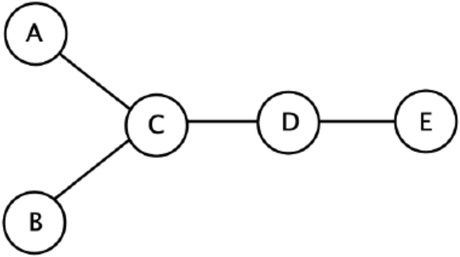
$$x_A : x_B : x_C : x_D : x_E : x_F \approx 1.00 : 0.87 : 0.76 : 0.41 : 0.41 : 0.35.$$



Highest betweenness centrality
Best closeness centrality
Highest eigenvector centrality
Highest degree centrality
a
b
c
d
e
f
g
h
i
j
k
l
m
n
o
p
q
r
s

Exercise 1

Determine the eigenvector centrality for all the nodes in the graph:



You may use a software in order to find the eigenvalues and vectors.

Random Walks and PageRank

Random Walks on a Graph

Definition (Random Walk on a Graph $G = (V, E)$)

This is a stochastic process $(X_t)_{t=0}^{\infty}$ with each $X_t \in V$ s.t.:

- Start with a node $v_0 = X_0$ chosen uniformly at random.
- If $X_t = i$ then X_{t+1} is a neighbour of i chosen uniformly at random from all its neighbours:

$$P_{ij} \; := \; \Pr(X_{t+1} = j | X_t = i) \; = \; \begin{cases} \frac{1}{\deg(i)}, & ij \text{ is a link} \\ 0, & \text{otherwise.} \end{cases}$$

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The matrix $P = (P_{ij}) \in \mathbb{R}^{N \times N}$ is called the transition matrix.

Note: $P = D^+ A_G$, where $D = \mathrm{diag}(\deg(1), \ldots, \deg(N))$.

$\rightarrow (D^+)_{ii} = 1/D_{ii}$ is $D_{ii} \neq 0$ and $(D^+)_{ii} = 0$ otherwise.

The resulting Markov chain

Let $\pi^{(t)} \in \mathbb{R}^N$ be the distribution of X_t, i.e., $\pi_i^{(t)} = \Pr(X_t = i)$. We have

$$\pi_i^{(t+1)} = \sum_{j=1}^{N} \Pr(X_t = j) \Pr(X_{t+1} = i | X_t = j) = \sum_{j=1}^{N} \pi_j^{(t)} P_{j,i}.$$

In other words, $\pi^{(t+1)} = P^\top \pi^{(t)}$.

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Note

- Define $\pi = \frac{1}{\mathrm{tr}(D)} D\mathbf{1}$ and recall $P = D^+ A_G$. So that

$$P^\top \pi \;=\; \frac{1}{\mathrm{tr}(D)} A_G D^+ D\mathbf{1} \;=\; \frac{1}{\mathrm{tr}(D)} A_G \mathbf{1} \;=\; \frac{1}{\mathrm{tr}(D)} D\mathbf{1} \;=\; \pi.$$

- We have $\pi_i = \frac{\deg(i)}{\sum_{j=1}^N \deg(j)}$ and so π is a probability distribution.

  (π defines the degree centrality!!)

- If $\pi^{(t)} = \pi$ then $\pi^{(s)} = \pi$ for all $s \geq t$; stationary distribution.

Eigenvalues of P

Note (Assume for simplicity all degrees positive; $D^{+} = D^{-1}$)

The transition matrix P is similar to a symmetric matrix:

$$P \;=\; D^{-1}A_G \;=\; D^{-1/2}D^{-1/2}A_G D^{-1/2}D^{1/2} = D^{-1/2}SD^{1/2}$$

and so it is diagonalizable. All eigenvalues lie in $[-1, 1]$.

Theorem (About the eigenvalues of P)

If G has no bipartite component, eigenvalues lie in $(-1, 1]$.

If G is connected, $\lambda = 1$ has multiplicity one.

Let $S = U\Lambda U^{\top}$ with U orthogonal. Let $\boldsymbol{u}_i$ be the i-th column of U. Then

$$S = \sum_{i=1}^{N} \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i^{\top} \qquad \text{and so} \qquad S^k = \sum_{i=1}^{N} \lambda_i^k \boldsymbol{u}_i \boldsymbol{u}_i^{\top} \underset{k\to\infty}{\longrightarrow} \boldsymbol{u}_1 \boldsymbol{u}_1^{\top},$$

where $\boldsymbol{u}_1$ is s.t. $S\boldsymbol{u}_1 = \boldsymbol{u}_1$.

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where $\boldsymbol{u}_1$ is s.t. $S\boldsymbol{u}_1 = \boldsymbol{u}_1$. It follows that $\boxed{P^k \longrightarrow \mathbf{1}\pi^{\top}}$.

Appendix: More formal arguments for $\lambda = -1$

Statement: P has eigenvalue $\lambda = -1$ if and only if G is bipartite.

Proof. $\boxed{\Leftarrow}$ If G is bipartite with partition $V = A \cup B$ define e_A to be a 0/1-vector with 1s on coordinates corresponding to A and 0s otherwise. It is a direct check that $P(e_A - e_B) = -(e_A - e_B)$.
$\boxed{\Rightarrow}$ There exists $\boldsymbol{x}$ such that $P\boldsymbol{x} = -\boldsymbol{x}$. Assume that G is connected. Otherwise apply the same argument to each connected component. The condition implies that for all $i \in V$

$$\sum_{j=1}^{N} P_{ij} x_j \;=\; \frac{1}{\deg(i)} \sum_{j \sim i} x_j \;=\; -x_i. \tag{1}$$

If $x_i = 0$ then (1) implies that $x_j = 0$ for $j \sim i$. Since G is connected, we would have $\boldsymbol{x} = 0$, which is impossible. We conclude, that $x_i \neq 0$ for all i. By (1), $\deg(i)|x_i| = |\sum_{j \sim i} x_j| \leq \sum_{j \sim i} |x_j|$. Summing over all i we get $\sum_i \deg(i)|x_i| \leq \sum_i \deg(i)|x_i|$ and hence the inequality must be equality for each i. This is only possible if $\forall i$ the sign of all x_j for $j \sim i$ is the same. Since all x_i are non-zero, this is only possible if G is bipartite. $\square$

Appendix: More formal arguments for $\lambda = 1$

Statement: If G is connected then $\lambda = 1$ has multiplicity one or, in other words, if $P\boldsymbol{x} = \boldsymbol{x}$ then $\boldsymbol{x} = c\mathbf{1}$ for some $c \neq 0$.

Proof. For every i, we have

$$\sum_{j=1}^{N} P_{ij} x_j \; = \; \frac{1}{\deg(i)} \sum_{j \sim i} x_j \; = \; x_i. \tag{2}$$

Suppose that $x_k = \max_i x_i$. The equation $\frac{1}{\deg(k)} \sum_{j \sim k} x_j = x_k$ implies that $x_j = x_k$ for all $j \sim k$. Using the fact that G is connected, we propagate this equality across the whole graph and so all the entries of $\boldsymbol{x}$ must be equal (and non-zero). $\square$

PageRank

Note

We define random walk on a directed graph in analogous way.

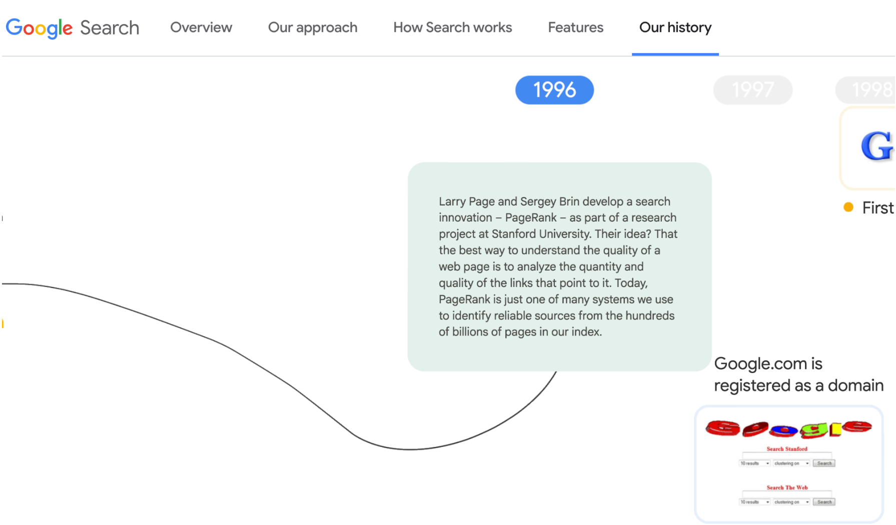
Algebraically more complicated as A_G is not symmetric and the eigenvalues are complex.

- Web graph $=$ directed network of pages and hyperlinks.
- Eigenvector centrality does not work directly in directed graphs with sinks or disconnected components.
- PageRank modifies the random walk with teleportation:

$$P_\alpha = \alpha P + (1-\alpha)\frac{1}{N}\mathbf{1}\mathbf{1}^T,$$

  where P is the transition matrix of the web, $\alpha \in (0, 1)$.
- Stationary distribution of P_α = PageRank vector.

https://www.google.com/search/howsearchworks/our-history/



- Solving for π = solving a huge eigenvector problem ($\sim 10^{10}$ nodes).
- Power iteration with $\alpha = 0.85$ converges in ~ 50 steps.

Computing Centrality in Python (NetworkX)

```
import networkx as nx

G = nx.karate_club_graph()

# Eigenvector centrality
eig = nx.eigenvector_centrality(G)
print(max(eig, key=eig.get))

# PageRank
pr = nx.pagerank(G, alpha=0.85)
print(max(pr, key=pr.get))
```

Karate club example: - Eigenvector centrality highlights the main hub (node 33). - PageRank is similar but also adapts to directed networks.

Conclusions

- Eigenvector centrality: nodes are important if linked to other important nodes.
- Perron–Frobenius ensures uniqueness and positivity of the principal eigenvector.
- PageRank extends the same idea to the Web via teleportation.
- Linear algebra (largest eigenvalue, eigenvector) is the foundation of centrality measures.