Universitat Pompeu Fabra Barcelona

Lecture 10 · Power laws and Hubs; Beyond Erdős–Rényi: Models for Real Networks

Networks, Crowds and Markets

Motivation: What ER misses

- In Lecture 9 we saw that real networks have:
  - high clustering,
  - heavy-tailed degree distributions,
  - and short average distances.
- Erdős–Rényi models explain only the last of these.
- Today we look closer into the power law and start building richer models that match all three.

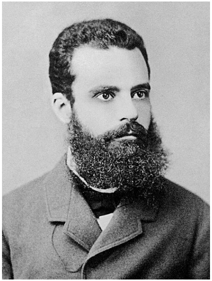
Today's Lecture

1. Power laws and hubs
2. Universality of power laws across networks.
3. Distances: small world vs ultra-small world.
4. Statistic random network models
5. Random network models with prescribed degree distribution.

Power laws and hubs

Historical roots: Pareto and the 80/20 law

Vilfredo Pareto (1848–1923), Italian economist, observed that income distribution in society is very uneven.



- Incomes followed a distribution with a heavy tail: a small fraction of people held most of the wealth.
- This became the well-known "80/20 rule": e.g. 20% of people control 80% of wealth.
- Similar patterns appear in many domains:
  - 80% of web links point to about 20% of webpages.
  - A small number of firms or banks control a large share of markets.
  - A few researchers or papers receive most citations.

Connection: Pareto's law is an early example of a *power law* in economics, closely related to what we now see in network degree distributions.

Power law: Discrete formalism

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We model the degree distribution as

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$\zeta(\gamma, k_{\text{min}})$ is the Hurwitz zeta function; for $k_{\text{min}} = 1$ it reduces to the Riemann zeta $\zeta(\gamma)$.

- The series converges if and only if $\gamma > 1$.
- In many real networks, empirical exponents satisfy $2 < \gamma \leq 3$.

First two moments

If $Z \sim (p_k)$ with $p_k = \dfrac{k^{-\gamma}}{\zeta(\gamma, k_{\min})}$ for $k \geq k_{\min}$, then

$$\mathbb{E}Z = \sum_{k \geq k_{\min}} k\, p_k = \frac{1}{\zeta(\gamma, k_{\min})} \sum_{k \geq k_{\min}} k^{-(\gamma-1)} = \frac{\zeta(\gamma-1, k_{\min})}{\zeta(\gamma, k_{\min})},$$

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The regime $2 < \gamma \leq 3$ is special:

- Since $\gamma - 1 > 1$, the mean exists.
- Since $\gamma - 2 \leq 1$, the variance does not!

(a very heavy-tailed distribution)

Power law: Continuum formalism

Sums like $\sum_{k \geq k_{\min}} k^{-\gamma}$ are hard to handle algebraically. For large networks (and large $k_{\min}$), approximate the sum by an integral:

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Define a density $p(k) = C\, k^{-\gamma}$ for $k \geq k_{\text{min}}$. Normalize:

$$1 = \int_{k_{\text{min}}}^{\infty} p(k) \, \mathrm{d}k = C \int_{k_{\text{min}}}^{\infty} k^{-\gamma} \, \mathrm{d}k = C \, \frac{k_{\text{min}}^{1-\gamma}}{\gamma - 1} \quad \Rightarrow \quad C = (\gamma - 1)\, k_{\text{min}}^{\gamma-1}.$$

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$C \approx 1/\zeta(\gamma, k_{\text{min}})$ with relative error $O(k_{\text{min}}^{-\gamma})$ for fixed $\gamma > 1$.

Extreme value of a power law: scaling of $k_{\max}$

With $p(k) = (\gamma - 1)\, k_{\min}^{\gamma-1}\, k^{-\gamma}$, the survival tail is

$$\mathbb{P}(K \geq k) = \int_k^{\infty} p(x)\, dx = \left(\frac{k_{\min}}{k}\right)^{\gamma-1}, \qquad k \geq k_{\min}.$$

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In a network with N nodes, we estimate the max-degree $k_{\max}$ by

$$\Pr(K \geq k_{\max}) \approx \tfrac{1}{N} \implies \left(\frac{k_{\min}}{k_{\max}}\right)^{\gamma-1} \approx \tfrac{1}{N}$$

$$k_{\max} \approx k_{\min}\, N^{1/(\gamma-1)}.$$

Notes.

- This captures the correct order; fluctuations are smaller-order.
- The same scaling holds for the discrete model up to constants.

Consequences of the $k_{\max}$ scaling

From $k_{\max} \approx k_{\min} N^{1/(\gamma-1)}$:

$$\begin{aligned}
\gamma = 2 \quad &\Rightarrow \quad k_{\max} \sim k_{\min} N \quad \text{(a single hub touches a linear fraction)} \\
2 < \gamma < 3 \quad &\Rightarrow \quad k_{\max} \sim k_{\min} N^{1/(\gamma-1)} \text{ sublinear but large} \\
\gamma = 3 \quad &\Rightarrow \quad k_{\max} \sim k_{\min} N^{1/2} \\
\gamma > 3 \quad &\Rightarrow \quad k_{\max} \text{ grows slowly; tails are lighter}
\end{aligned}$$

Path lenghts in Scale-Free Networks

Average path length in random networks

Let $d(u, v)$ be the distance between two vertices and $\mathbb{E}[d]$ the average distance across all pairs.

- In Erdős–Rényi graphs with mean degree c fixed,

$$\mathbb{E}[d] \sim \frac{\ln N}{\ln c}.$$

- In scale-free networks with degree tail $p_k \sim k^{-\gamma}$,

$$\mathbb{E}[d] \sim \begin{cases} \text{constant}, & \gamma = 2, \\ \ln\ln N, & 2 < \gamma < 3, \\ \frac{\ln N}{\ln\ln N}, & \gamma = 3, \\ \ln N, & \gamma > 3. \end{cases}$$

Idea: The scaling of $\mathbb{E}[d]$ reflects how large the biggest hub can grow, $k_{\max} \approx k_{\min} N^{1/(\gamma-1)}$, and how efficiently hubs act as shortcuts.

Case $\gamma = 2$: hub-and-spoke regime

Here $\mathbb{E}[d] = O(1)$.

- From $k_{\max} \sim k_{\min} N^{1/(\gamma-1)}$, we get $k_{\max} \sim N$: one hub connects to almost all nodes.
- The graph becomes star-like (*hub–and–spoke* structure). Any two peripheral nodes connect via the hub in at most two steps.
- Therefore $\mathbb{E}[d]$ remains bounded independently of N.
- Networks with $\gamma = 2$ are extremely centralized and fragile to hub removal.

Case $2 < \gamma < 3$: ultra-small world

- Here $k_{\text{max}} \sim k_{\text{min}} N^{1/(\gamma-1)}$ grows faster than any power of $\ln N$ but slower than N.
- A few very large hubs act as shortcuts, giving

$$\mathbb{E}[d] \sim \ln\ln N \quad (\text{“ultra-small world”}).$$

- The mean degree $\mathbb{E}[\text{deg}]$ is finite but $\mathbb{E}[\text{deg}^2] = \infty$: variance diverges, so hubs dominate connectivity.
- Most empirical scale-free networks (social, technological, biological) fall in this range.

Case $\gamma = 3$: critical point

- The largest degree scales as $k_{\max} \sim N^{1/2}$.
- The second moment $\mathbb{E}[\deg^2]$ stops diverging but is still large.
- This produces a slower, logarithmically corrected growth:

$$\mathbb{E}[d] \;\sim\; \frac{\ln N}{\ln\ln N}.$$

- Paths are longer than in the $\gamma < 3$ case but still shorter than in Erdős–Rényi graphs.

Case $\gamma > 3$: small-world regime

- Both mean and variance of deg are finite: hubs are limited in size.
- $k_{\max} \sim N^{1/(\gamma-1)}$ grows slowly, producing no global shortcuts.
- The average distance recovers the classic small-world scaling:

$$\mathbb{E}[d] \sim \ln N.$$

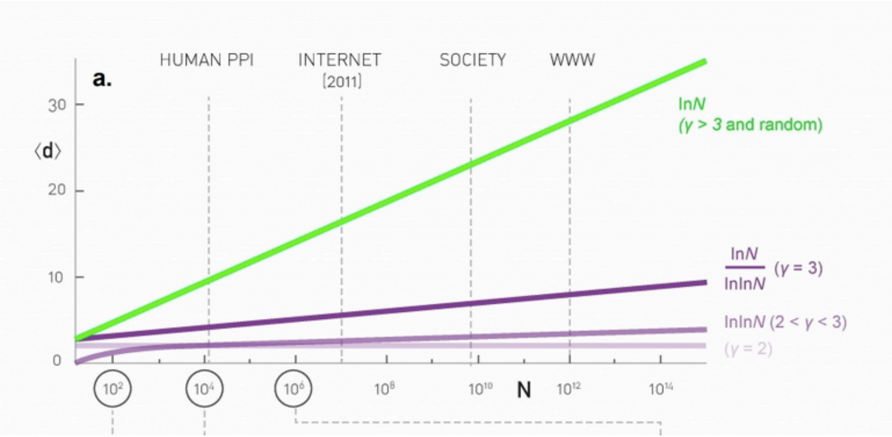
- This regime behaves similarly to Erdős–Rényi graphs in terms of average distance.

When $\gamma < 2$: nonphysical limit

- Then $1/(\gamma - 1) > 1$, so

$$k_{\text{max}} \sim k_{\text{min}} \, N^{1/(\gamma-1)}$$

  grows faster than N.
- This would require nodes of degree larger than the entire network — impossible in a simple graph.
- Moreover $\mathbb{E}[\text{deg}]$ diverges: even the mean degree is infinite.
- $\Rightarrow$ Infinite scale-free networks with $\gamma < 2$ cannot exist; finite networks must have an effective cutoff.



Note that for large networks the difference in average degrees between the four regimes is much larger than for small networks.

Conclusions

In summary, the effects on distances in scale-free networks are:

- They **shrink average path lengths**. Most scale-free networks of practical interest are “ultra-small”, because hubs act as bridges linking many low-degree nodes.
- They **change the scaling of $\mathbb{E}[d]$ with system size**: the smaller the exponent γ, the shorter the distances between nodes.
- Only for $\gamma > 3$ do we recover the $\mathbb{E}[d] \sim \ln N$ scaling — the *small-world* behavior characteristic of Erdős–Rényi graphs.

Next: we explore richer models that explain how such networks emerge.

Need for more sophisticated models

Erdős–Rényi: clean benchmark for randomness in networks.

- Degrees: Binomial $\to$ Poisson in sparse regime, sharply concentrated (Hoeffding).
- Sharp thresholds: giant component at $p \sim 1/N$, full connectivity at $p \sim (\log N)/N$.

Analytic power: every property can be studied precisely—gives language for thresholds, asymptotics, and "with high probability" results.

But realism is limited:

- Clustering $\mathbb{E}[C_v] = p \to 0$ as $N \to \infty$ (in the sparse regime).
- Degree distribution thin-tailed: no hubs or communities.
- Real social, financial, and web networks are way more structured.

This motivates a study of other random graph models.