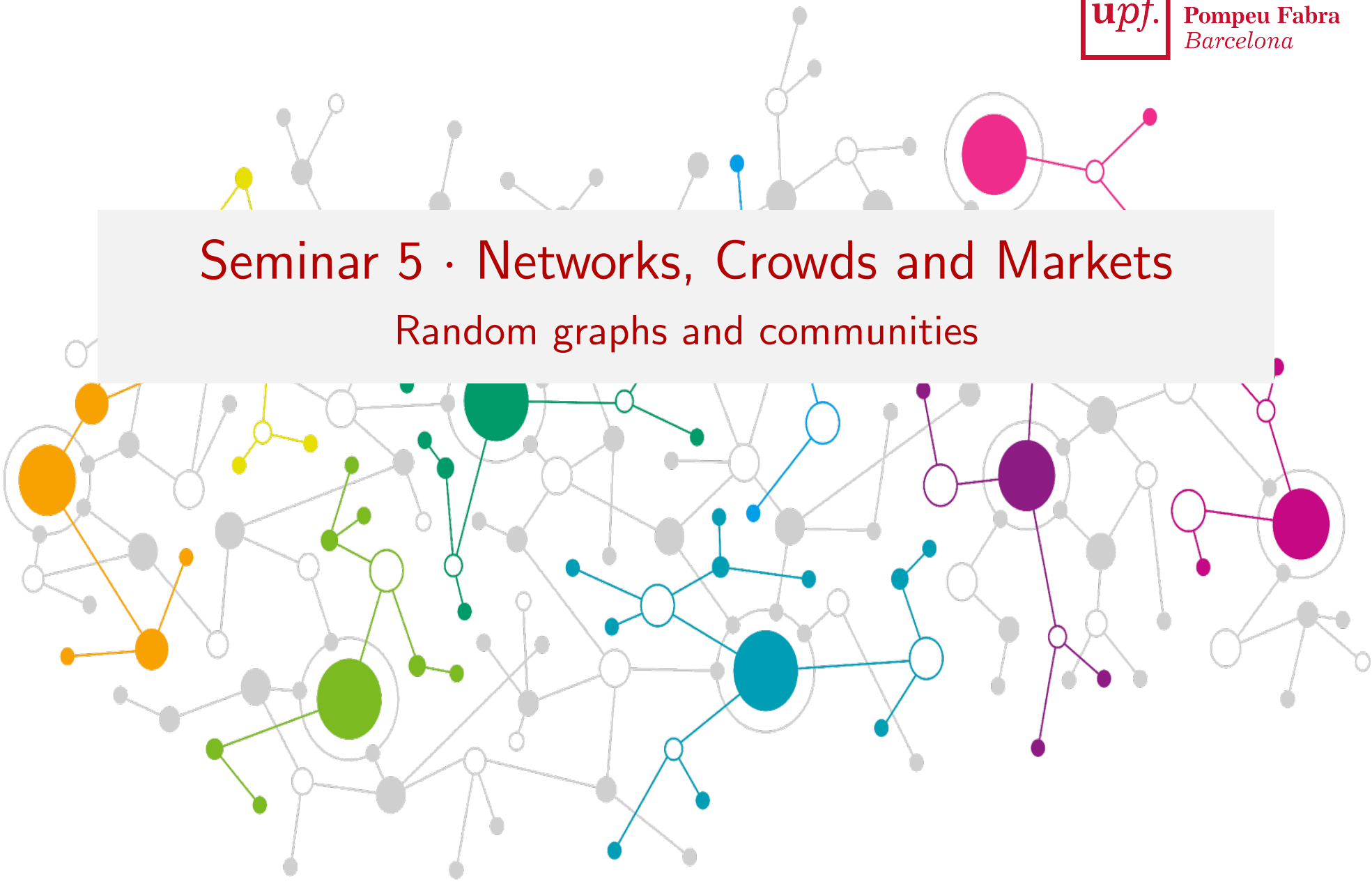


upf. Universitat Pompeu Fabra Barcelona

Seminar 5 · Networks, Crowds and Markets

Random graphs and communities

Configuration model and preferential attachment

In the first part we will experiment a bit with the two models.

See the corresponding colab.

Sampling the configuration model

We have N nodes and degree sequence $(k_1, \ldots, k_N)$. Let each node i own k_i *stubs* (half-edges).

Sampling algorithm (uniform over all pairings):

1. Make a list of all $2L = \sum_i k_i$ stubs. Label each stub by its node owner.
2. While unpaired stubs remain:
   - 2.1 Select a stub uniformly at random.
   - 2.2 Choose its partner uniformly at random among the remaining stubs.
   - 2.3 Connect their owners with an edge (i, j).
   - 2.4 Remove both stubs.
   - 2.5 Repeat.
3. The resulting multigraph is one sample from the configuration model.

Note: loops (i, i) and parallel edges may appear. They are rare for large sparse networks.

Exercise: Correctness of the algorithm

This is a reformulated version of this result.

Show that the algorithm described above generates a multigraph with the right degree sequence. Show that the probability of each pairing is uniform.

Solution: Why the pairing is uniform

Pairing of a set of $2L$ elements: a collection of 2-element subsets that are disjoint and whose union gives the whole set.

Let $2L = \sum_i k_i$ be the number of stubs. There are $(2L-1)!! = (2L-1)(2L-3)\cdots 1$ pairings between these stubs.

Proof of uniformity.
Record the sampled stubs as an ordered sequence $(s_1, s_2;\ s_3, s_4;\ \ldots;\ s_{2L-1}, s_{2L})$. Any fixed ordered sequence has probability

$$\frac{1}{2L}\cdot\frac{1}{2L-1}\cdot\frac{1}{2L-2}\cdots\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{(2L)!}.$$

A single unordered pairing can be realized by $L!$ orders of the L pairs and 2 orders inside each pair: $2^L L!$ sequences total. Hence

$$\Pr(\text{a given pairing}) = \frac{2^L L!}{(2L)!} = \frac{1}{(2L-1)!!},$$

the same for every pairing.

Pairings do not define the multigraph uniquely

As pointed out by Calixta in the first seminar group: pairings do not uniquely define multigraphs. For example, if we have two degree two vertices A, B with stubs A_1, A_2, B_1, B_2 then pairings $\{A_1, B_1\}, \{A_2, B_2\}$ and $\{A_1, B_2\}, \{A_2, B_1\}$ both encode the double edge between A and B.

The distribution over multigraphs is in general not uniform.

Depending on a multigraph, there may be a different number of pairings leading to it. On the next slide we discuss an example.

Counterexample

Take four nodes A, B, C, D with degrees $(1, 2, 2, 1)$. The stubs are $A_1, B_1, B_2, C_1, C_2, D_1$ with 15 pairings.

The simple graph $A - B - C - D$ corresponds to four pairings of the form $\{A_1, B_i\}, \{B_{3-i}, C_j\}, \{C_{3-j}, D_1\}$ for $i, j = 1, 2$.

The simple graph $A - C - B - D$ also corresponds to four pairings.

Double edge between B, C and an edge between A, D is defined by two pairings $\{B_i, C_j\}, \{B_{3-i}, C_{3-j}\}, \{A_1, D_1\}$ with $i = j$ or $i \neq j$.

The graph with B, C having loops and an edge between A, D is given by a single pairing.

The graphs with B having a loop and $A - C - D$ is given by two pairings.

The graphs with C having a loop and $A - B - D$ is given by two pairings.

The probability of each multigraph $= \frac{\#\text{pairings}}{15}$.

Exercise: Degree distribution in preferential attachment

Consider the **preferential attachment model** with $m = 1$.

Let d_t denote the degree of the initial vertex at time t.

(a) What is the distribution of d_5?

(b) What are $\Pr(d_t = 1)$ and $\Pr(d_t = t - 1)$?

(c) Find the exact expression for

$$\mathbb{E}[\, d_{t+1} - d_t \mid d_t = k \,].$$

(d) Show that

$$\mathbb{E}[\, d_t \,] = \frac{(2t-3)!!}{2^{t-2}(t-2)!} = \frac{(2t-3)!!}{(2t-4)!!}, \qquad \text{where } n!! = n(n-2)(n-4)\cdots.$$

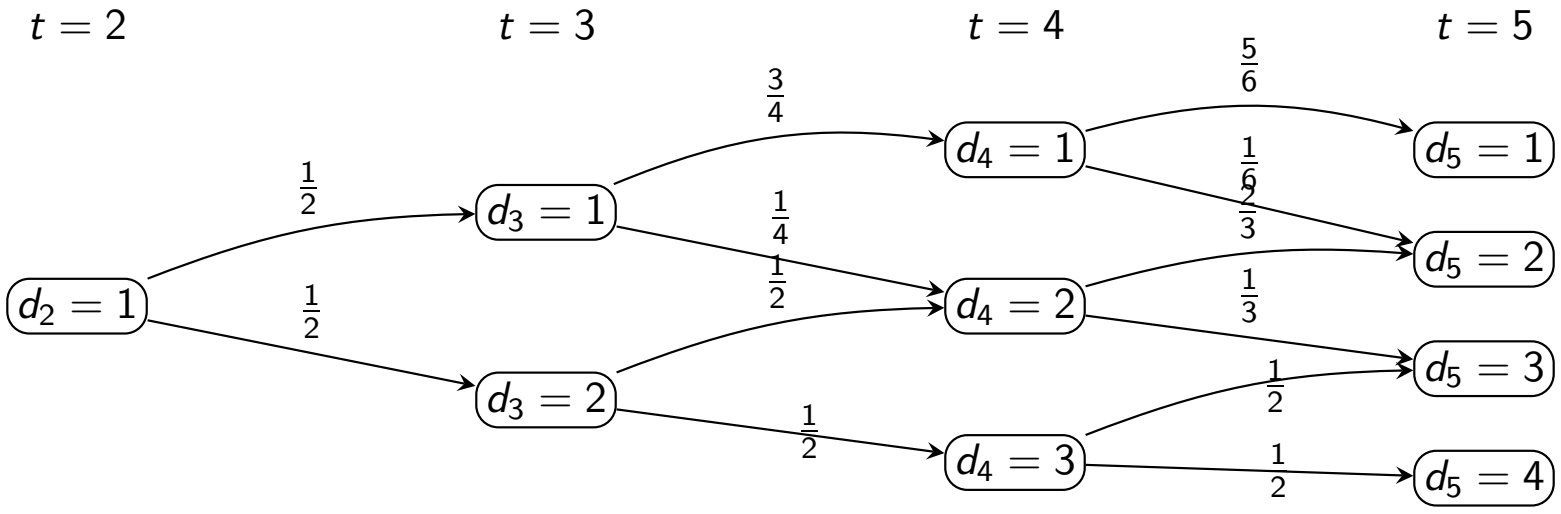
Hint: express the degree growth as a random multiplicative process and use the recurrence relation for $\mathbb{E}[d_t]$.

Solution sketch (a) – corrected transitions

Setup. At time t, total degree is $2(t-1)$. If $d_t = k$, then

$$\Pr(d_{t+1} = k+1 \mid d_t = k) = \frac{k}{2(t-1)}, \qquad \Pr(d_{t+1} = k \mid d_t = k) = 1 - \frac{k}{2(t-1)}.$$

Transition diagram (from $t = 2$ to $t = 5$).



Distribution at $t = 5$:

$$\Pr(d_5 = 1) = \tfrac{5}{16}, \quad \Pr(d_5 = 2) = \tfrac{5}{16}, \quad \Pr(d_5 = 3) = \tfrac{1}{4}, \quad \Pr(d_5 = 4) = \tfrac{1}{8}.$$

Solution (b)–(d): Endpoints and mean

Endpoints (all $t \geq 2$):

$$\Pr(d_t = 1) = \prod_{s=2}^{t-1}\Big(1 - \frac{1}{2(s-1)}\Big) = \frac{(2t-4)!!}{(2t-3)!!},$$

$$\Pr(d_t = t-1) = \frac{(t-2)!}{(2t-4)!!} = \frac{1}{2^{t-2}}.$$

Drift and mean.

$$\mathbb{E}[d_{t+1} - d_t \mid d_t = k] = \frac{k}{2(t-1)} \quad \Rightarrow \quad \mathbb{E}[d_{t+1}] = \Big(1 + \frac{1}{2(t-1)}\Big)\mathbb{E}[d_t],$$

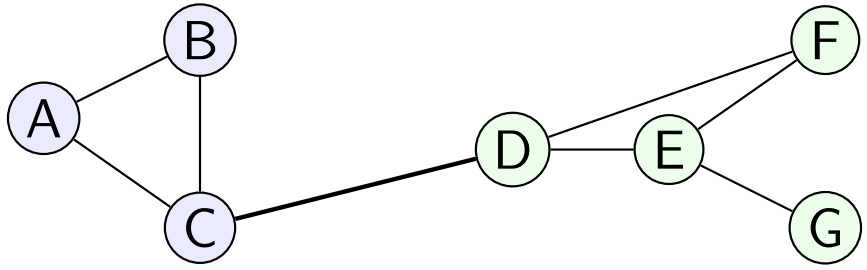
with $d_2 = 1$. Hence

$$\mathbb{E}[d_t] = \prod_{s=2}^{t-1}\Big(1 + \frac{1}{2(s-1)}\Big) = \frac{(2t-3)!!}{2^{t-2}(t-2)!} = \frac{(2t-3)!!}{(2t-4)!!}.$$

Check at $t = 5$: $\mathbb{E}[d_5] = \frac{7!!}{6!!} = \frac{105}{48} = \frac{35}{16} \approx 2.1875$, which matches the diagram.

Exercise: Detecting communities

Graph: two dense parts joined by a single bridge.



(a) Check whether $\{A, B, C\}$ and $\{D, E, F, G\}$ form strong or weak communities.

(b) Using the similarity matrix on the next slide, perform **the full average-linkage hierarchical clustering**, starting from the seven singleton clusters. List the merges in order until only two clusters remain.

(c) Compare the modularity for the two partitions

$$\mathcal{P}_1 = \{\{A, B, C\}, \{D, E, F, G\}\}, \quad \mathcal{P}_2 = \{\{A, B, C\}, \{D, E, F\}, \{G\}\}.$$

Solution: The similarity matrix

Neighborhoods (including the node itself):

$$B_A = \{A, B, C\}, \quad B_B = \{A, B, C\}, \quad B_C = \{A, B, C, D\},$$
$$B_D = \{C, D, E, F\}, \quad B_E = \{D, E, F, G\}, \quad B_F = \{D, E, F\}, \quad B_G = \{E, G\}.$$

	A	B	C	D	E	F	G
A	1.00	1.00	1.00	0.33	0	0	0
B	1.00	1.00	1.00	0.33	0	0	0
C	1.00	1.00	1.00	0.50	0.25	0.33	0
D	0.33	0.33	0.50	1.00	0.75	1.00	0.50
E	0	0	0.25	0.75	1.00	1.00	1.00
F	0	0	0.33	1.00	1.00	1.00	0.50
G	0	0	0	0.50	1.00	0.50	1.00

Off-diagonal entries are topological overlap similarities s_{ij}. For example, the bridge C–D gives $s_{CD} = 0.5$, the triangle D–E–F gives $s_{DF} = s_{EF} = 1$, and the leaf G attached to E yields $s_{EG} = 1$ and $s_{DG} = s_{FG} = 0.5$.

Solution: Detecting communities (a)

(a) Strong vs weak communities.
Degrees:

$\deg(A) = 2,\ \deg(B) = 2,\ \deg(C) = 3,\ \deg(D) = 3,\ \deg(E) = 3,\ \deg(F) = 2,\ \deg$

Group $S_1 = \{A, B, C\}$:

- A: $\deg_{\text{in}} = 2,\ \deg_{\text{out}} = 0,$
- B: $\deg_{\text{in}} = 2,\ \deg_{\text{out}} = 0,$
- C: $\deg_{\text{in}} = 2,\ \deg_{\text{out}} = 1.$

So S_1 is a strong community.
Group $S_2 = \{D, E, F, G\}$:

- D: $\deg_{\text{in}} = 2,\ \deg_{\text{out}} = 1,$
- E: $\deg_{\text{in}} = 3,\ \deg_{\text{out}} = 0,$
- F: $\deg_{\text{in}} = 2,\ \deg_{\text{out}} = 0,$
- G: $\deg_{\text{in}} = 1,\ \deg_{\text{out}} = 0.$

Thus S_2 is also a strong community.

Solution: Detecting communities (b) (1/3)

(b) Full average-linkage clustering.
We start from the partition:

$$\{\{A\},\{B\},\{C\},\{D\},\{E\},\{F\},\{G\}\}.$$

Step 1. Largest similarity $= 1.00$. Pairs with similarity 1:

$$(A,B),(A,C),(B,C),(D,F),(E,F),(E,G).$$

To break ties we merge an arbitrary pair:

$$\{A\},\{B\} \longrightarrow \{A,B\}.$$

Step 2. Recompute cluster similarities. Cluster $\{A,B\}$ has similarity 1.00 with C:

$$s(\{A,B\},C) = 1.00.$$

Thus:

$$\{A,B\},\{C\} \longrightarrow \{A,B,C\}.$$

Solution: Detecting communities (b) (2/3)

Step 3. Continue on the right side. Remaining high similarities on the right part:

$$(D, F),\ (E, F),\ (E, G).$$

We first merge D and F:

$$\{D\}, \{F\} \longrightarrow \{D, F\}.$$

Step 4. Merge E and G.
Now the clusters are

$$\{A, B, C\},\ \{D, F\},\ \{E\},\ \{G\}.$$

Among pairs of singleton clusters, the largest similarity is

$$s(E, G) = 1.00.$$

Hence

$$\{E\}, \{G\} \longrightarrow \{E, G\}.$$

Solution: Detecting communities (b) (3/3)

Step 5. Merge the two right-hand clusters.

Current clusters:

$$\{A, B, C\},\ \{D, F\},\ \{E, G\}.$$

Average-linkage similarity between the two right-hand clusters is

$$s(\{D, F\}, \{E, G\}) = \frac{s_{DE} + s_{DG} + s_{FE} + s_{FG}}{4} = \frac{0.75 + 0.50 + 1.00 + 0.50}{4} = 0.6875,$$

which is larger than their similarities with $\{A, B, C\}$. Therefore we merge

$$\{D, F\}, \{E, G\} \longrightarrow \{D, E, F, G\}.$$

Final two clusters:

$$\{A, B, C\} \qquad \text{and} \qquad \{D, E, F, G\}.$$

Solution: Detecting communities (c)

(c) Modularity comparison.
Using

$$Q(\mathcal{P}) = \frac{1}{2m} \sum_{i,j} \left(A_{ij} - \frac{k_i k_j}{2m} \right) \mathbf{1}\{c_i = c_j\}, \qquad m = 8,$$

we obtain

$$Q(\mathcal{P}_1) \approx 0.37, \qquad Q(\mathcal{P}_2) \approx 0.30.$$

Thus $\mathcal{P}_1$ has higher modularity.