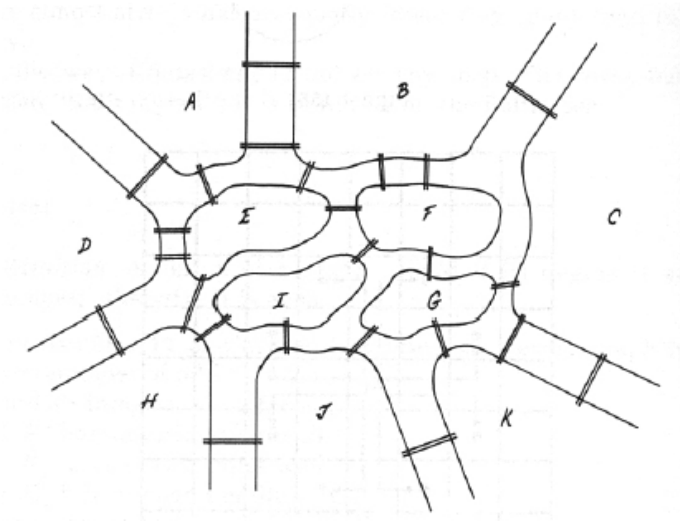
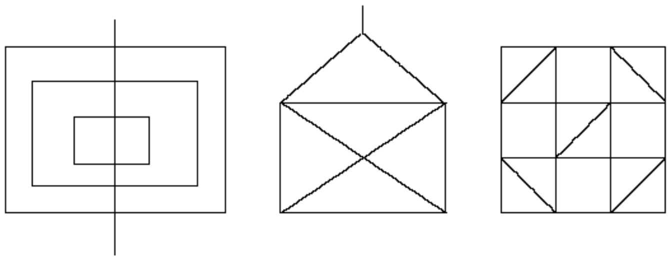
Königsberg Bridge and Eulerian Drawings

1. Is it possible to cross all bridges exactly once in the map below (you may revisit locations, but you cannot reuse a bridge)? If so, show such a route. If not, explain why.

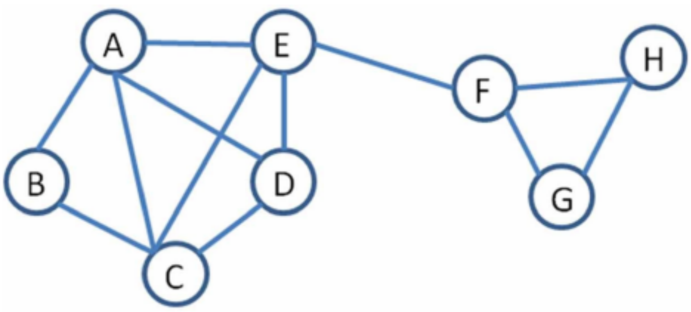


2. For each of the drawings below, determine whether it can be traced in a single stroke without lifting your pencil and without retracing a line. Justify your answer.



Introduction to graph Theory

1. Given the following Graph G:

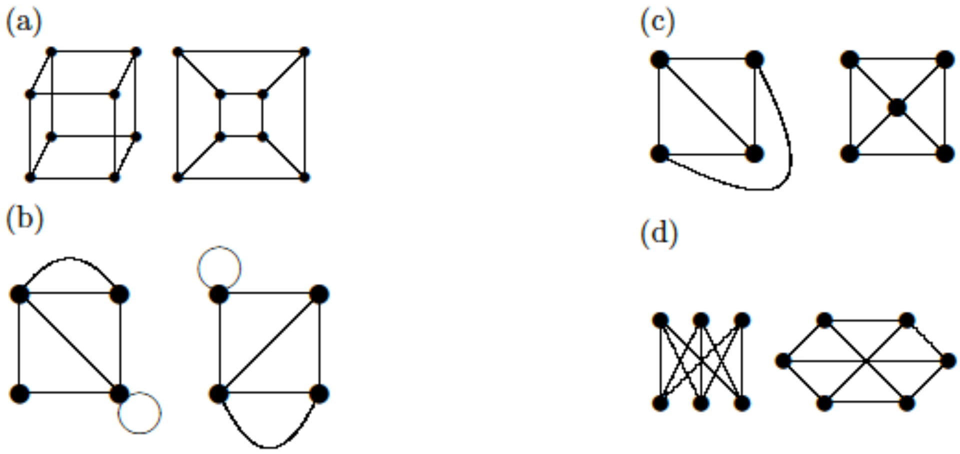


Find:

(a) The set of vertices (nodes) of the graph.

(b) The set of edges (links) of the graph.
(c) The adjacency matrix of the graph.
(d) The list of nodes of even degree.
(e) The degree sequence.
(f) The average degree of G
(g) The degree distribution.
(h) The diameter and eccentricity of the graph.

2. Study whether the following graphs are isomorphic or not:

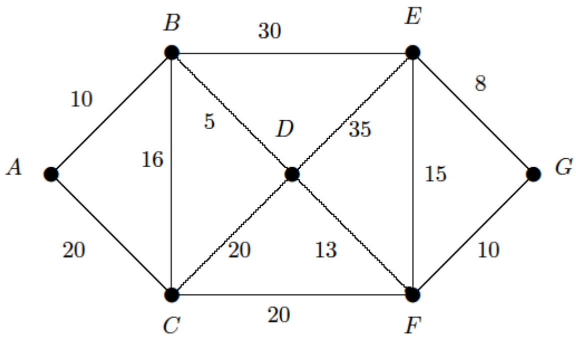


3. Given a graph with the following degree sequence $\{1, 1, 1, 1, 2, 3, 5\}$.

(a) Draw a possible graph with the given sequence.
(b) Determine the number of edges of the graph.
(c) Determine the average degree of the graph.
(d) Determine its degree distribution.

4. For each $i = 1, \ldots, 6$, draw all graphs without loops (if possible) having the following degree sequence: $\{1, \ldots, i\}$.

5. Find the adjacency matrix of the following graph:



6. Draw the graphs corresponding to the following adjacency matrices.

(a) Simple graph:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

(b) Multigraph:

$$B = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

7. Each pair of matrices below represents possible adjacency matrices of a graph G and its complement G^c. Decide for which pairs this is possible.

(a) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

8. A bipartite graph G divides its vertex set into two disjoint parts of sizes M and N.

(a) Determine the maximum number of edges that such a graph can have.

(b) Let G' be the complete simple graph on the same $M + N$ vertices. Compute the ratio

$$r = \frac{|E(G)|}{|E(G')|}$$

when G has the maximum possible number of edges.

9. Given a bipartite graph on N vertices (without specifying the partition sizes), determine the maximum number of edges it can have.

10. Draw an $r-$regular graph of 12 nodes for $r = 1, 2, 3$.

11. Prove that if a graph is regular of odd degree, then it has even order (number of vertices).

12. Given a complete graph K_n, for what values of n do we obtain a path graph P_n when deleting one edge of K_n?

13. Let G be a 4-regular simple graph.

(a) What is the maximum number of edges that can be removed by deleting three vertices from G?

(b) Give an example where this maximum is achieved or explain why it cannot be.

14. Find out whether the complement of a regular graph is regular, and whether the complement of a bipartite graph is bipartite. If so, prove it; if not, give a counterexample.

15. Consider a graph G of size n and m edges. Let v be a node and e an edge of G. Give the size and the number of edges of G^c, $G - v$ and $G - e$.

16. Let G be a simple graph on $n \geq 6$ vertices. Prove that either G or its complement G^c contains a vertex of degree at least 3.

17. Given a node u from a connected graph G of 2023 vertices. We know that $k_u = 2022$, prove that G^c is not connected.

18. Show that if two graphs G, H having set of vertices V_G, V_H are connected and $V_G \cap V_H \neq \emptyset$, then $G \cup H$ is connected.

19. Show that if a graph G is disconnected, then its complement G^c must be connected. Give an explicit example of a graph on four vertices for which both G and G^c are connected.

20. Find the diameter of the following graphs.

a) K_n

b) $K_{r,s}$

c) C_n

d) P_n

e) If r, R are the radius (the minimum local diameter of any node) and diameter of a connected graph G, show that $R \leq 2r$.

f) If $(G_n)_{n\geq 1}$ is a sequence of undirected connected graphs of increasing size and bounded diameter, then the maximum undirected degree of $(G_n)_{n\geq 1}$ is unbounded.

Random Networks

1. Show that in an Erdős–Rényi random network $G(N, p)$, when N is large and the average degree $\overline{\deg}(G) = p(N-1)$ is small compared to N, the binomial degree distribution can be approximated by a Poisson distribution.

Remark: This is the regime often called the *sparse limit.* You may use the standard Taylor expansion of $\ln(1-p)$ to simplify expressions.

2. The transition from a disconnected to a connected random network occurs when the expected number of isolated nodes drops to approximately one. Show that this happens when

$$p = \frac{\ln N}{N}.$$

3. Consider an Erdős–Rényi random network with $N = 8000$ nodes, where each possible edge is present independently with probability $p = 0.5 \times 10^{-3}$.

(a) Compute the expected number of edges $\langle L \rangle$.

(b) Determine the critical probability p_c at which a giant component first emerges.

(c) For $p = 0.5 \times 10^{-3}$, find the approximate number of nodes N_c required for the network to be connected with high probability.

(d) Compute the average degree $\overline{\deg}(G)$ and estimate the typical distance between two nodes in the connected regime.

4. Suppose we have a random network with $\overline{\deg}(G) = 6.3$ and $N = 192{,}244$.

(a) Compute the network density, defined as the ratio of existing edges to possible edges.

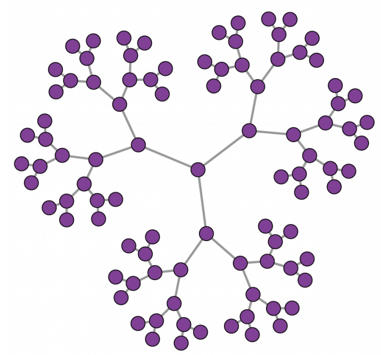
(b) Identify the connectivity regime of the network (subcritical, critical, or supercritical) and explain why.

(c) Estimate the average path length.

5. Consider a connected random network with $N = 10^6$ and average degree $\overline{\deg}(G) = 5$.

(a) Find the linking probability p.
(b) Determine the expected number of edges.
(c) Estimate the probability that a randomly chosen node has degree at least three.
(d) Identify the connectivity regime of the network.
(e) Approximate the diameter of the network.

6. A Cayley tree is a symmetric tree that starts from a central node of degree K. Each node at distance $d < D$ from the center has degree K, and nodes at distance D (the leaves) have degree 1.



Example of a Cayley tree with $K = 3$ and $D = 5$.

(a) Show that the number of nodes reachable in t steps ($t < D$) from the central node is

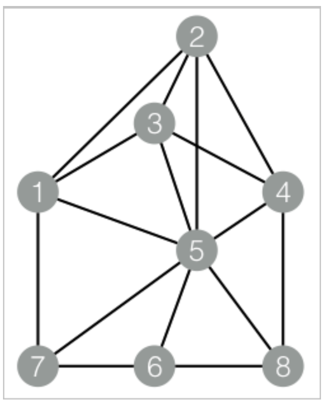
$$N_t = \frac{K(K-1)^t - 2}{K-2}.$$

(b) Compute the degree distribution of the network.
(c) Find the diameter $d_{\max}$.
(d) Express $d_{\max}$ in terms of the total number of nodes N.

7. Consider a population of N blue and N red nodes. Each pair of same-color nodes is linked with probability p, and each pair of nodes of different colors is linked with probability q. A network is said to be *snobbish* if $p > q$.

(a) Compute the expected degree of a blue node within the blue subnetwork, and its total expected degree in the full network.
(b) Determine the minimal conditions on p and q for the network to be connected with high probability.
(c) Discuss whether such networks can still display the small-world property when $p \gg q$.

8. For the graph below, compute the clustering coefficients of nodes 2, 5, and 7.



Scale-Free Networks and BA Model

21. Prove that for Scale-Free Networks the maximun degree of a node may be found using:

$$K_{max} = k_{min} N^{\frac{1}{\gamma-1}}$$

Hint: "The Largest Hub"section seen in lecture 6 might be useful.

22. Calculate the expected maximum degree k_{max} for the networks listed in the following table:

	N	L	$\overline{\deg}(G)$	γ_{in}	γ_{out}	γ
Actor Network	702388	$3 \cdot 10^7$	83.71			2.12
WWW	325729	$1.5 \cdot 10^5$	4.60	2.1	2.45	
Mobile Phone Network	36595	91826	2.51	4.69	5.01	

23. Given a SFN with $N = 10^9$ nodes and $\gamma = 3.3$, $k_{min} = 1$, determine:

(a) Its degree distribution.
(b) The value of average $\overline{\deg}(G)$.
(c) The expected number of links.
(d) The standard deviation.
(e) The probability of having a node with maximum 3 links.
(f) The number of hubs that the network has (degree of a hub $k \geq 10^7$)
(g) Is the network sparse?

Remark: You may derivate your results using the continuum formalism.

24. Given a SFN with $N = 5000$ nodes and $\gamma = 2.4$, $k_{min} = 1$, determine:

(a) Its degree distribution (in both formalisms).
(b) The value of average $\overline{\deg}(G)$.
(c) The probability of having a node exactly 5 links.

25. What is the "rich-get-richer"phenomena? Find information about it, explain the concept and give a context why this phenomena emerges in the Barabási Albert Model.

26. Suppose that we have a network with the following degree exponential distribution $p(k) = Ce^{-\lambda k}$, where C is the constant of normalization. Assuming that k_{min} is the minimum degree that we may find in the network.

(a) Determine the constant C.
(b) Find the maxim number of links connected to a single node K_{max} using the same assumption we did in class when determinating K_{max} for Scale-Free networks.

Remark: You may derivate your results using the continuum formalism.

27. The degree distribution p_k expresses the probability that a randomly selected node has k neighbors. However, if we randomly select a link, the probability that a node at one of its ends has degree k is $q_k = Akp_k$, where A is a normalization factor.

(a) Find the normalization factor A, assuming that the network has a power law degree distribution with $2 < \gamma < 3$, with minimum degree k_{min} and maximum degree k_{max}.

(b) In the configuration model q_k is also the probability that a randomly chosen node has a neighbor with degree k. What is the average degree of the neighbors of a randomly chosen node?

(c) Calculate the average degree of the neighbors of a randomly chosen node in a network with $\gamma = 2.3$, $k_{min} = 1$ and $k_{max} = 1000$. Compare the result with the average degree of the network, $\langle k \rangle$.

(d) How can you explain the "paradox"of (c), that is a node's friends have more friends than the node itself? Find in the literature the friendship paradox and give a propper explanation.

28. What are the 2 key points of the BA model?

Centrality measures

29. Compute the degree centrality, the closeness centrality and the betweeness centrality for the following network:

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31. Compute the general eigenvector centrality for the next network using the iterative method (use 3 iterations).

32. A matrix A is a nilpotent if there exist $t > 0$ such that $A^t = 0$. Show an example of a graph of size 5 such that its adjacency matrix is nilpotent and compute for this graph the Katz-Bonacich centrality measure setting the attenuating factor $\lambda = 0.5$.

Linear Algebra

33. Given the matrix:

$$A = \begin{pmatrix} -1 & 4 & 0 \\ 0 & 8 & 6 \\ 3 & 0 & 9 \end{pmatrix}$$

(a) Determine its eigenvalues and eigenvectors.

(b) Does A diagonalize? If that is the case, find a diagonal matrix D and a matrix P such that $A = PDP^{-1}$.

34. Given the matrices:

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{pmatrix}$$

Determine A^{30} and B^{500}.

35. Given the matrix:

$$A = \begin{pmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

(a) Diagonalize the matrix.

(b) Find 3 ortonormal eigenvectors.